Voting Dilemmas: Is Democracy a Mathematical Farce?

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"It's not the voting that's democracy; it's the counting."
–Tom Stoppard

Outline

Presidential Election, 2016

- Many voters are not excited about either major-party nominee
- Voting for other parties (Libertarian, Green, Communist, The Rent is Too Damn High) is effectively throwing your vote away
- Wish there were a more effective way of expressing preferences

A Better Way of Voting–Ranking Candidates

- No run-off votes, all candidates on ballot at the same time
- Voters *rank* all of the candidates instead of just voting for top choice
- Greater information allows for better/more choices in elections

Outline

Preference Ballots

- Instead of voting for one person, each voter gives a ranking of the options.
- Suppose four candidates: Adam, Bernie, Clinton, and Donald (A, B, C, and D).
- A sample ballot looks like:

1st-	B
2nd-	D
3rd-	C
4th-	A

Preference Schedules

- Preference ballots are compiled and arranged into preference schedules.
- Sample preference schedule:

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Means 14 voters voted A B C D, 10 voters voted C B D A, etc.

Outline

Plurality Method

- Whoever has the most first-place votes wins the election. Do not need a majority.
- This is effectively the system we have for most elections in the U.S. (with run-offs in some cases)
- For

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

The Plurality winner is A, with 14 first-place votes. Next highest is C, with 11.

Borda Count Method

- Each candidate is given points for each ranking: 4 points for 1st place, 3 for 2nd, 2 for 3rd, 1 for 4th.
- Points are added up and the candidate with the most points wins the election.
- This is the system used for college football polls, also for other sports awards (Heisman trophy) and hiring decisions.

Borda Count Example

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

- Candidate A has 14 first place votes, 23 4th place votes. Total for A: $(14\times\ 4) + (23\times\ 1)$=79 points.
- Candidate B: $(4 \times 4) + (24 \times 3) + (9 \times 2) = 106$ points.
- Candidate C gets 104 points, D gets 81 points.
- The Borda Count winner is Candidate B.

Instant Runoff Method

- Election has multiple 'run-off' elections. After first vote, candidate with lowest number of first-place votes is eliminated- their votes are added to next candidate.
- Process is repeated, with one candidate eliminated in each round.
- Run-off ends when one candidate has a majority of first-place votes.
- Used in some municipal elections in the US (esp. Bay area), also parliament in Australia.
- Allows for 3rd party candidates without causing a spoiler effect (Here's looking at you, Ralph Nader).

Instant Runoff Example

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

- Candidate B has lowest number of first place votes and is eliminated from preference schedule in round 1. New preferences are:

Number of voters	14	10	8	4	1
1st choice	A	C	D	D	C
2nd choice	C	D	C	C	D
3rd choice	D	A	A	A	A

Instant Runoff Example, cont.

Number of voters	14	10	8	4	1
1st choice	A	C	D	D	C
2nd choice	C	D	C	C	D
3rd choice	D	A	A	A	A

- D now has 12 first place votes, so C is eliminated in round 2 (only 11 first-place votes).

Number of voters	14	10	8	4	1
1st choice	A	D	D	D	D
2nd choice	D	A	A	A	A

- Candidate D wins the election in round 3.

Method of Pairwise Comparisons

- Each candidate is compared head-to-head with each other candidate. If a candidate wins a head-to-head matchup, they get a point.
- After all head-to-head matchups are tabulated, candidate with highest number of points wins the election.

Pairwise Comparison Example

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

- A vs B: B wins 23-14. 1 point for B.
- A vs C: C wins 23-14. 1 point for C.
- A vs D: D wins 23-14. 1 point for D.
- C vs B: C wins 19-18. 1 point for C.
- C vs D: C wins 25-12. 1 point for C.
- D vs B: B wins 28-9. 1 point for B.
- C wins the election with a total of 3 points.

Summary of Methods

- With Plurality Method, A wins.
- With Borda Count, B wins.
- With Instant Runoff, D wins.
- With Pairwise Comparison, C wins (in fact, C beats every other candidate in a head-to-head competition).
- So, selection of winner of ranked election can depend on the choice of vote-counting method.
- Note: There are many *other* voting methods (Schulze method, Approval Voting, Range Voting, Borda Count with Elimination, etc)

Outline

Majority Criterion

- Majority Criterion: If a candidate wins a majority of the first-place votes, that candidate should win the election.
- The Electoral College fails the Majority Criterion (much to Al Gore's chagrin)
- Borda Count fails the majority criterion. All other methods satisfy it.
- In the past, have had college football teams ranked #1 even though they received fewer first-place votes than another team.

Majority Criterion Failure Example

Number of voters	10	5	4
1st choice	A	C	C
2nd choice	C	B	A
3rd choice	B	A	B

- A gets 30+5+8= 43 points
- C gets 20+27= 47 points
- C wins the Borda count, even though a majority chose A for first place.

Pairwise Winner Criterion

- Pairwise Winner Criterion: If there is a candidate who beats all other candidates in a head-to-head competition, that candidate should win the election.
- Other than pairwise comparison, all other voting methods fail this criterion.
- Our original election exemplifies those failures; though C won the pairwise comparison, she would lose if we used any of the other methods.

Independence of Irrelevant Alternatives (IIA)

- Independence of Irrelevant Alternatives criterion states that should the following occur:
  1. Election with a winner (say, C) and candidate B is a loser.
  2. B drops out of the race, so we re-tally by eliminating B.
- Then the IIA criterion states that C should still be the winner.
- Pairwise comparison fails this criterion, and it is the only one that pairwise comparison fails.
- Electoral College also fails IIA, as Ralph Nader showed in 2000

IIA failure Example (part 1)

NFL Draft example:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- When do pairwise comparisons: A gets 3 points, 2.5 for B, 2 for C, 1.5 for D, 1 for E.
- So A wins the election.

IIA failure Example (part 2)

NFL Draft example: C decides at last minute to not enter the draft.

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	B	D	D	E
2nd choice	D	A	A	A	A	A	D
3rd choice	B	D	D	D	B	E	B
4th choice	E	E	E	E	E	B	A

- When do pairwise comparisons: A gets 2 points, 2.5 for B, 1.5 for D, 0 for E.
- So B wins the re-tally, even though an 'irrelevant' alternative was eliminated.

Monotonicity Criterion

- Monotonicity Criterion: Suppose the following occur:
  1. An election is held and a candidate (say, C) is the winner.
  2. Some preference ballots are changed, but only in favor of C.
- Then the Monotonicity criterion states that C should still be the winner.
- The instant runoff method fails monotonicity, but all of the others satisfy it.

Monotonicity Criterion failure Example, pt 1

Straw poll results for Olympic host:

Number of voters	7	8	10	4
1st choice	A	B	C	A
2nd choice	B	C	A	C
3rd choice	C	A	B	B

- Round One: B is eliminated.
- C wins in round 2.

Monotonicity Criterion failure Example, pt 2

Straw poll results get out, and people in the last column change their choice *in favor of the winner, C*:

Number of voters	7	8	14
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

- Round One: A is eliminated.
- B wins in round 2.
- By changing their vote in favor of C, the voters actually caused C to lose!

Other Fairness Criterion

- Participation Criterion- The addition of more votes favoring a winner should not make that winner lose.
- Clone-proof Criterion- Addition of "similar" candidates does not affect the outcome of an election.
- Later-no-harm Criterion- Additional ranking or approval of less-preferred candidates will not cause a more-preferred candidate to lose.
- There are at least 12 more on Wikipedia alone, though many are similar.
- None of our current methods satisfy ALL criteria, so we need to find another method that *does*.

Arrow's Impossibility Theorem

- Arrow's Impossibility Theorem states:
  " With three or more candidates and any number of voters, there does not exist a voting system that satisfies all of the fairness criterion and always produces a winner."
- Hence, democracy is a farce. We should all go home and cry.

With two caveats

Caveat One:

- Arrow's Impossibility Theorem only applies to certain kinds of voting methods that involve *ranking*
- Other voting methods (e.g. approval, range voting) do not fall under the impossibility theorem
- But they have problems of their own.

Caveat Two:

- Maybe fairness violations don't occur in real elections very often, if at all.

Outline

The big question

- Mathematically, there are problems with all voting systems
- Question: How often do these problems crop up in real-world elections?
- Specific question: How often do you have a Monotonicity Anomaly in Instant Runoff elections?
- Theoretical research (using fabricated "general" data) indicates that monotonicity violations should happen a lot–as high as 15% of the time
- Or very rarely–as low as 0.03% of the time
- Depends on whose research you believe

The data

- Biggest problem is little data (Australia and Ireland are not sharing)
- Luckily, all of the data from San Francisco is freely available online
- Can use available voting data to check for Monotonicity anomalies

The research

- David Naylor, Nick Zayatz and I worked on finding monotonicity anomalies
- Method:
  1. Run IRV election to find winner
  2. Change ballots by moving that winner up in certain ballots
  3. Rerun election to see if changes result in a different winner
- Analyzed over 100 elections: 62 had 3 or more candidates, 36 were competitive, 24 had potential Monotonicity Violations (by inspection)

Basic method

- Change one whole column of preference schedule:

# voters	7	8	10	4
1st	A	B	C	A
2nd	B	C	A	C
3rd	C	A	B	B

→

# voters	7	8	14
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

- Program found no monotonicity anomalies in any of the available data.

Better method

- Check all *combinations* of swaps
- "Advanced" program found no monotonicity anomalies. Then we realized it didn't actually check all combinations.
- Made new program that DID check all combinations (All-swaps program)
- All-swaps took too long to run–David estimated it would take 25 times the age of the universe to run the program for large data sets.
- We did not have 25 times the age of the universe

Approximation of Runs

$$\text{Simple monotonicity check} \quad : \quad \left[\sum_{j=1}^{n}(P_j - 1)\right] + 1$$

$$\text{Advanced check} \quad : \quad \left[\sum_{i=1}^{n}\sum_{j=1}^{i}(P_j - 1)\right] + 1$$

$$\text{All-swaps check} \quad : \quad \prod_{j=1}^{n} P_j$$

- As many as 10^{169} runs for All-swaps program on the Burlington, VT, data set.

Methods to speed up the program

- Depth-first approach: "Weighting" the columns in some way to find the most important switches.
- Result: Still took too long, got no results
- Greedy approach: First check columns that move votes from second-place finisher to winner.
- Result: No monotonicity anomalies from greedy approach
- Top-three approach: Run IRV election until only three candidates left, then do all-swaps
- Result: Fairly fast to run, no anomalies found

Even bigger problem

- Some monotonicity anomalies would not be found by all-swaps, even if it DID work.

Number of voters	22	21	14	15	30
1st choice	A	A	C	C	B
2nd choice	B	C	A	B	C
3rd choice	C	B	B	A	A

- Swapping *all* BCA votes to ABC $\rightarrow$ A wins in first round.
- Swapping two BCA votes to ABC $\rightarrow$ B drops out, gives 28 votes to C, C wins.

Smart check

- Look for gaps between candidate votes as they drop out.
- If moving winner up can change order that candidates drop out, could result in monotonicity anomaly.

Number of voters	22	21	14	15	30
1st choice	A	A	C	C	B
2nd choice	B	C	A	B	C
3rd choice	C	B	B	A	A

- In example above, need two votes to change order, so program should only swap two votes

Top Three Smart Check

- Program runs IRV election to top three candidates, then calculates the "gap" in votes between 2nd and 3rd place (n votes)
- Program swaps winner higher in $n+1$ ballots, then checks to see if winning candidate changes
- Top three smart check program found a monotonicity anomaly in one set of data, mayoral race in Burlington, VT.
- No other anomalies found in any other IRV elections

Summary of results

- Real-world anomalies do not seem to be as prevalent as theoretical data would indicate, but do exist.
- So all hope is not lost.
- But, though programs have not found other anomalies, some anomalies might exist

Current focus and future research

- Future goals:
  - Total smart check (may take too long), or at least Top Four
  - No show paradox anomalies
  - Approval voting comparison
  - Independence of Irrelevant alternatives anomalies

Questions?

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Outline

Recently in the news

- IRV is used in many municipal elections in the US.
- Recently there has been a backlash with some municipalities choosing to revoke IRV and go back to previous system.
- Interesting case in Burlington, Vermont election in 2009.

Burlington Election, 2009

- Initially 6 candidates, 2 had virtually no support, a third had a fair bit less than other three candidates. Lower three all get eliminated.
- In second round 3 candidates left were Bob Kiss (Progressive), Kurt Wright (Republican), and Andy Montroll (Democrat).
- First-place votes were W-3294, K-2981, M-2554
- Montroll is eliminated, 2/3 of his votes go to Kiss who wins in the third round.
- In a head-to-head, though, Montroll would have beat Kiss by a solid 8 percent.
- Controversy ensues, Burlington revokes IRV in 2010 by a vote of 52 percent to 48 percent.

Other voting methods - Easy

- Approval Voting- voter just says "Approve" or "Disapprove" for each candidate. All "approves" are tallied, candidate with most votes wins.
- Range voting- voter gives each candidate a score from 0 to 10. Like approval voting, but allows voters to discriminate more finely between those they approve and disapprove.
- There are 14 voting methods listed on Wikipedia. Paper by Warren Smith analyzes 40 different methods - many are similar, though.

Other voting methods - Complicated

- Schulze Method- Candidates are compared head-to-head.
- A candidate who loses to another can still be considered to "beat" that candidate if there is a chain of preferences that allows it.
- Eliminate "rejects". This is a candidate A who is beat in a head-to-head by candidate B and there is no chain by which A can beat B.
- Process does NOT always result in a winner- some sort of runoff procedure is often necessary to winnow down to a single winner.
- Runoff procedure- Find "weakest pairwise defeat" and turn it into a tie, then repeat the process.

References

- Wikipedia "Voting System" page.
- Voting Systems paper by Paul Johnson (eventually to be a part of a textbook by Saul Stahl)
- Two websites that hate each other: fairvote.org (website promoting IRV) and rangevoting.org (website promoting range voting)