Test 2 - Math Thought

Dr. Graham-Squire, Spring 2016

9:54
10:16
22

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

DIRECTIONS

(1) Don't panic.

(2) Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.

(3) You are required to do the first 4 questions on the test. For questions 5 through 8, you only need to do three of the questions. It is fine if you do all four of the questions 5-8, though–I will grade them all and just give you the points for the top 3 scores.

(4) There is a take-home portion of the test as well, and it is two problems.

(5) Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul.

(6) Make sure you sign the pledge above.

(7) Number of questions = 8 in-class, 2 take-home. Total Points = 50.

(1) (6 points) Prove that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

whenever n is a positive integer.

Base Case: $n=1 \Rightarrow \frac{1}{1\cdot 3} \stackrel{?}{=} \frac{1}{2(1)+1} = \frac{1}{3}$

Induction: Suppose $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

Want to prove true for $P(k+1)$.

$$\underbrace{\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$

$$= \frac{k+1}{2(k+1)+1}$$

$\boxed{}$

(2) (4 points) Let A and B be sets. Is $(A \times B)^c = A^c \times B^c$? If so, prove it. If not, give a counterexample.

<u>Not</u> always equal! Consider $U = \{1, \dots, 10\}$

$A = \{1, 2, \dots, 5\}$, $B = \{1, 2, \dots, 5\}$. Then $(1,6) \notin A \times B$,

So $(1,6) \in (A \times B)^c$. But $1 \in A$, so ~~$(1,6) \notin$~~ $1 \notin A^c$

$\Rightarrow (1,6) \notin A^c \times B^c$.

$\underset{=(x,y)}{u} \in A^c \times B^c \Rightarrow x \in A^c, y \in B^c \Rightarrow x \notin A, y \notin B$

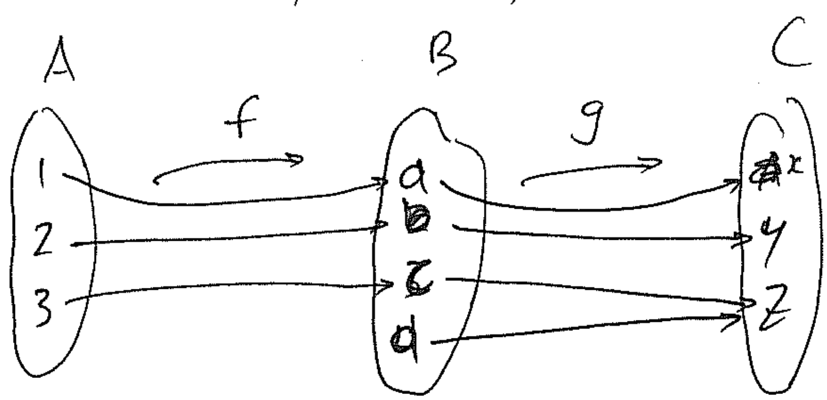
$x \notin A \Rightarrow u \notin A \times B \Rightarrow u \in (A \times B)^c$

$u \in (A \times B)^c \Rightarrow \underset{=(x,y)}{u} \notin (A \times B) \Rightarrow x \notin A$ <u>or</u> $y \notin B$ up to 0.5

To be in $A^c \times B^c$ need $x \notin A$ <u>and</u> $y \notin B$, so

$(A \times B)^c \notin A^c \times B^c$.

(3) (3 points) Give an example of functions $g : B \to C$ and $f : A \to B$ such that $g \circ f : A \to C$ is an injection, but $g : B \to C$ is NOT an injection. (Note: you can use "real" functions that involve algebraic expressions, but arrow diagrams of proofs are also okay. No matter what, make sure you clearly show what the functions are, and what their domain/codomains are.)



g not 1-1 b/c $g(c) = g(d)$ but $c \neq d$.

$g \circ f$ is 1-1, though.

+1.5 for each condition

-0.5 for other crazy things (not a function, etc)

(4) (4 points) Is the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $g(x, y) = 5x + 10y$ an onto (surjective) function? If so, prove it. If not, find a counterexample.

✓✓

Not onto! $g(x,y) = 5x + 10y = 5(x+2y)$,

so 5 divides everything in the image of

✓ g. Thus ~~[illegible]~~ $1 \in \mathbb{Z}$, but there is no

(x,y) such that $g(x,y) = 1$

b/c if $\frac{5x+10y}{5} = \frac{1}{5}$

$$\underbrace{x+2y}_{\text{Must be an integer b/c } x,y \in \mathbb{Z}} = \frac{1}{5}$$

✓

not an integer

So impossible.

$\Rightarrow$ Not surjective.

up to +2 for proof of surjective.

For the next four problems, I will only give you the top *three* scores. So you can choose to do only three (and skip one problem), or you can do all of them, and I will grade all four and drop the lowest score of the four.

(5) (6 points) Consider the distributive property $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. Prove one subset inclusion for this equation (you do NOT need to prove both subset inclusions, only one).

$(\subseteq)$ ✓

Let $x \in (A \cap B) \cup C$. Then by def. of union,

$x \in A \cap B$ or $x \in C$. ✓

- Case 1: if $x \in C$, then $x \in A \cup C$ (by property of union) and $x \in B \cup C$, so $x \in (A \cup C) \cap (B \cup C)$ by def. of intersection. ✓ ✓

- Case 2: if $x \in A \cap B$, then $x \in A$ and $x \in B$ by def. of intersection. $x \in A \Rightarrow x \in A \cup C$ by property of union, $x \in B \Rightarrow x \in B \cup C$ by same prop. ✓ Then $x \in A \cup C$ and $x \in B \cup C \Rightarrow$

$x \in (A \cup C) \cap (B \cup C)$ by def. of intersection.

$\blacksquare$

(6) (6 points) Let A, B and C be sets. Prove that $A \cup (B \cap C) \subseteq (\overline{B - (A \cup C)})^c$ Note: a Venn diagram may help your thinking, but it is not sufficient as a proof. You can also use the attached list of set properties, if you would like.

Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$ ✓

• Case 1: $x \in A$. Then $x \in A \cup C$ (by prop. of union)

✓✓ and $x \notin B - (A \cup C)$ (def of set difference)

$\Rightarrow x \in (B - (A \cup C))^c$ (def. of complement.

✓✓ · Case 2: $x \in B \cap C$. Then $x \in B$ and $x \in C$, by def. of intersection. $x \in C \Rightarrow x \in A \cup C$ (prop. of union)

$\Rightarrow x \notin B - (A \cup C)$ (def. of $-$)

$\Rightarrow x \in (B - (A \cup C))^c$ (def of complement)

$\blacksquare$

Or

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(B - (A \cup C))^c = (B \cap (A \cup C)^c)^c$

$= \cancel{(B \cap (A^c \cap C^c))}$

$= B^c \cup (A \cup C)$

$= B^c \cup A \cup C$

$A \cup (B \cap C) \overset{?}{\subseteq} A \cup C \cup B^c$

Case 1: $x \in A \Rightarrow x \in A \cup C \cup B^c$ (prop. of Union)

Case 2: $x \in (B \cap C) \Rightarrow x \in C \Rightarrow x \in A \cup C \cup B^c$ (prop. of union)

$\blacksquare$ ✓

up to +3 for using set properties!

(7) (6 points) Using the ordered pair definition of function, we say that a set of ordered pairs $\{(a,b) \mid a \in A \text{ and } b \in B\}$ represents a function $f : A \to B$ if the set has the following two properties:

(a) For all $x \in A$, there exists an ordered pair (x,b) in the set.

(b) For all $x \in A$, if there exist ordered pairs (x,b) and (x,c) in the set, then $b = c$.

Use the definitions above to explain why a function must be both injective (one-to-one) and surjective (onto) in ordered to have an inverse function $f^{-1} : B \to A$ defined.

for $f^{-1}: B \to A$ to be a function, you must be able to reverse the ordered pairs above and still have it fit the properties (1) and (2).

Thus the ordered pairs are (b,a), and

$\Rightarrow$ property (1) says for all $b \in B$, there must be an ordered pair $(b,a) \Leftrightarrow \exists\, a \in A$ s.t. $f(a)=b$ $\Leftrightarrow f$ is onto.

$\Rightarrow$ property (2) says $\forall\, b \in B$, if (b,x) and (b,z) and in the set, then $x = z$. But (b,x) in the set means $(x,b) \in f \Leftrightarrow f(x)=b$. Similarly we have $f(z)=b$.

Thus prop. (2) says if $f(x)=f(z)$, then $x=z$. Which is the definition of injective!

Max of 3 w/o ordered pairs.

(8) (6 points) Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f : A \to C$ is an injection, then $f : A \to B$ is an injection.

To prove $f: A \to B$ is 1-1, we assume that $f(a) = f(b)$, and we wish to prove that $a = b$.

If $f(a) = f(b)$, then applying g to be sides we have

$$g(f(a)) = g(f(b))$$

$$\Leftrightarrow (g \circ f)(a) = (g \circ f)(b)$$

$$\Rightarrow a = b$$ b/c $g \circ f$ is 1-1 by assumption

Extra Credit (up to 2 points) Choose 0.5 or 2 points. If you choose 0.5, you are guaranteed to get one-half of an extra credit point. If you choose 2, and four or more *other* students in the class choose 2, then everyone who chose 2 gets *no* extra credit. If fewer than 5 students in the class choose 2, then everyone who puts a 2 gets 2 extra credit points.

2	0.5
III	~~IIII~~ II

List of set properties:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A - B = A \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$
- $A \subseteq B$ if and only if $B^c \subseteq A^c$

Test 2 - Math Thought

Dr. Graham-Squire, Spring 2016

Take-home portion of the test

10:19
10:27
8

Name: Key

⇒ between 30 and 45 min.

I pledge that I have neither given nor received any unauthorized assistance on this take-home portion of the exam.

(*signature*)

DIRECTIONS

(1) Don't panic.
(2) Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
(3) You should do both questions to the best of your ability.
(4) Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul. It should go without saying (but I am saying it here anyway) that you should not speak to anyone else about any of the questions on this portion of the test until our next class (Wednesday) when it is due.
(5) Make sure you sign the pledge above.
(6) Number of questions = 2. Total Points = 10.

(1) (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 1$.

(a) Calculate an expression for the composition of f with itself. That is, find an expression for $(f \circ f)(x)$.

(b) Calculate an expression for the composition of f with $f \circ f$. That is, find an expression for$((f \circ f) \circ f)(x)$.

(c) Let f^n be defined as the composition of f with itself n times. Thus the answer for (a) could be written as $(f \circ f)(x) = f^2(x)$, and the answer for (b) would be $((f \circ f) \circ f)(x) = f^3(x)$. Calculate more compositions (if necessary) to find a general expression (in terms of x, n and possibly some numbers) for $f^n(x)$.

(d) Use induction to prove that your answer for (c) is correct.

(a) $f(f(x)) = f(2x+1) = 2(2x+1) + 1 = 4x+3$

(b) $(f \circ f)(f(x)) = 4(2x+1) + 3 = 8x+4+3 = 8x+7$

(c) $f^4(x) = 8(2x+1) + 7 = 16x+15$

$\Rightarrow$ seems like $f^{(n)}(x) = 2^n x + (2^n - 1)$

(d) Base step: $f^{(1)}(x) = 2^1 x + (2^1 - 1) = 2x+1 = f(x)$ ✓

Inductive step: Suppose $f^{(k)}(x) = 2^k x + (2^k - 1)$

$$f^{(k+1)}(x) = 2^k(2x+1) + (2^k - 1)$$
$$= 2^{k+1}x + 2^k + 2^k - 1$$
$$= 2^{k+1}x + 2(2^k) - 1$$
$$= 2^{(k+1)}x + 2^{(k+1)} - 1$$

$\blacksquare$

(2) (5 points) Use induction and the distributive property

$$(A_1 \cap A_2) \cup B = (A_1 \cup B) \cap (A_2 \cup B)$$

to prove that for all $n \in \mathbb{Z}^+$, $n \geq 2$, if $A_1, A_2, \ldots, A_n$ and B are sets then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

Base Case: $n=2$. Then $(A_1 \cap A_2) \cup B = (A_1 \cup B) \cap (A_2 \cup B)$ is true by

Inductive Step: Suppose

$(A_1 \cap A_2 \cap \cdots \cap A_k) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B)$

Then

$$(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) \cup B$$

$$= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}) \cup B$$

$$= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cup B) \cap (A_{k+1} \cup B) \quad \text{by base case}$$

$$= (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B) \quad \text{by inductive Hyp.}$$

$\blacksquare$