Test 1 - Math Thought

Dr. Graham-Squire, Spring 2016

3:19
3:48
29 ⇒ too long.

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

DIRECTIONS

(1) Don't panic.

(2) Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.

(3) Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul.

(4) Make sure you sign the pledge above.

(5) Number of questions = 10. Total Points = 45.

T F

(1) ($\cancel{5}$ 4 points) Find truth values for P, Q, R and S to show that $[(P \to Q) \to R] \to S$ and $P \to [Q \to (R \to S)]$ are not logically equivalent.

False

~~Let P = False = Q = R and S = F.~~

$$\cancel{\text{Then } [(P \to Q) \to R] \to S = [(F \to F) \to F] \to F = [T \to F] \to F = F \to F \text{ is True}}$$

~~and P~~

Let $P = Q = S = F$ and $R = T$.

Then $[(P \to Q) \to R] \to S$ and $P \to [Q \to (R \to S)]$

$[T \to T] \to F$ | $F \to [F \to (F \to F)]$

$T \to F$ | $F \to [F \to F]$

is False | $F \to T$

is True

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(2) (~~5~~ points) A sequence of real number $\{x_1, x_2, x_3, \dots\}$ is called a BooYaa sequence provided that for each positive real number a, there exists a positive integer N such that for all $z \in \mathbb{Z}^+$, if $z > N$, then $|x_z| < a$.

(a) Use mathematical notation to express what it means to be a BooYaa sequence.

(b) Use words to careful explain what it means for a sequence to NOT be a BooYaa sequence.

2 (a) Booyaa $\Leftrightarrow (\forall a \in \mathbb{R}^+)(\exists N \in \mathbb{Z}^+) \mid (\forall z \in \mathbb{Z}^+)$

$$[(z > N) \rightarrow |x_z| < a]$$

(b) Negation is $(\exists a \in \mathbb{R}^+)(\forall N \in \mathbb{Z}^+)(\exists z \in \mathbb{Z}^+)$

$$[(z > N) \wedge |x_z| \geq a]$$

2 There exists $a \in \mathbb{R}^+$ such that for all $N \in \mathbb{Z}^+$ there exists $z \in \mathbb{Z}^+$ such that $z > N$ and $|x_z| \geq a$.

-0.5 for small issue with negation of $\Rightarrow$

-1 for big " " " " "

(3) (5 points)

(a) Let the domain be $\mathbb{R}$ =real numbers. Are the following statements true or false? Are they logically equivalent? Explain why or why not.

(i) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(x + y - z = 0)$

(ii) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})(x + y - z = 0)$

(i) is ~~True~~ False. For all x, if you choose a y, then $z = x+y$, so it won't work for all z.

(ii) Is true. For any x and y, Let $z = x+y$ and it works.

So they are not logically equivalent since one is true, other is false.

add 0.5 if good answer for log. equiv., don't take off.

(4) (5 points) Let n be an integer. Use the *definitions* of even and/or odd to prove the following:

$$n + 3 \text{ is even} \leftrightarrow n^2 + 3 \text{ is even.}$$

0.5

$(\rightarrow)$ $n+3$ is even $\Rightarrow n+3 = 2\ell$ for $\ell \in \mathbb{Z}$

$\Rightarrow n = 2\ell - 3$

$\Rightarrow n^2 = (2\ell-3)^2 = 4\ell^2 - 12\ell + 9$

4

$\Rightarrow n^2+3 = 4\ell^2 - 12\ell + 12$

$\Rightarrow n^2+3 = 2(2\ell^2 - 6\ell + 6)$

which is even by definition.

0.5

$(\leftarrow)$ n^2+3 even $\rightarrow n+3$ is even

Proof by contrapositive: Suppose $n+3$ is odd.

Then $n+3 = 2\ell + 1$ for $\ell \in \mathbb{Z}$

WTP: n^2+3 is odd.

4

$\Rightarrow n = 2\ell - 2$

$\Rightarrow n^2 = 4\ell^2 - 8\ell + 4$

6+1

$\Rightarrow n^2+3 = 4\ell^2 - 8\ell + 7$

$\Rightarrow n^2+3 = 2(2\ell^2 - 4\ell + 3) + 1$

is odd by def. $\blacksquare$

(7) (5 points) Suppose that $a, b, c \in \mathbb{Z}$ and $c \neq 0$. Prove that if ac divides bc, then a divides b.

$ac / bc \iff (ac)n = bc$ for $n \in \mathbb{Z}$

$\iff \frac{acn}{c} = \frac{bc}{c}$ ← can divide by c b/c $c \neq 0$

$\iff an = b$

$\iff$ a divides b by definition.

−0.5 if no b/c $c \neq 0$.

(5)

~~8~~ (5 points) For the following statement/proof, you must choose one of two options:

(a) The statement is false, and thus the proof is false. In this case, you must say how/where the proof is false and then give a counterexample or explanation to show how the statement is false.

(b) The statement is true, but the proof is false and/or poorly written. In this case, you must say how/where the proof is false or what about it is poorly written, and then write a correct proof.

The statement: "If $n \not\equiv 2 \pmod{10}$, then $n^2 \not\equiv 4 \pmod{10}$."

Proof. Proof by contrapositive. Suppose that $n \equiv 2 \pmod{10}$. Then by the definition and properties of equivalence modulo 10, we have

$$
\begin{array}{rcll}
n & = & 2+10k & \text{for some } k \in \mathbb{Z} \\
\rightarrow n^2 & = & (2+10k)^2 & \text{(squaring both sides)} \\
\rightarrow n^2 & = & 4+40k+100k^2 & \text{(math)} \\
\rightarrow n^2 & = & 4+10(4k+10k^2) & \text{(factoring)} \\
\rightarrow n^2 & \equiv & 4 \pmod{10} & \text{(definition of congruence mod 10)}
\end{array}
$$

Thus we have shown that if $n \equiv 2 \pmod{10}$, then $n^2 \equiv 4 \pmod{10}$, so by the contrapositive we have "If $n \not\equiv 2 \pmod{10}$, then $n^2 \not\equiv 4 \pmod{10}$," as desired. □

Proof is false b/c contrapositive should be assuming that $n^2 \equiv 4 \pmod{10} \Rightarrow n \equiv 2 \pmod{10}$, and In the proof they do the converse of this.

The statement is false, because if $n \equiv 8 \pmod{10}$ then $n^2 \equiv 64 \pmod{10} \equiv 4 \pmod{10}$, so $n \not\equiv 2 \pmod{10}$, but $n^2 \equiv 4 \pmod{10}$ for $n \equiv 8 \pmod{10}$

For questions 7 to 10, you only need to answer 3 out of the four questions. If you answer all of them, that is fine–I will grade all of them and give you the highest 3 scores from the 4 questions.

(6)

(7) (5 points) Is the following statement true or false? If true, prove it. If false, find a counterexample.

For all integers x and y, $(x+y)^2 \equiv (x^2+y^2) \pmod 2$.

✓✓

means $(x+y)^2 = x^2+y^2+2k$ for some $k \in \mathbb{Z}$.

True! $(x+y)^2 = x^2+2xy+y^2$

✓✓✓ $= (x^2+y^2)+2(xy)$

$\Rightarrow (x+y)^2 \equiv (x^2+y^2) \pmod 2$

$xy \in \mathbb{Z}$ b/c of closure of integers under mult.

-1 for assuming what want to prove

(but if work is right, then 4/5)

(8) (5 points) Prove that if m is an integer, then 3 divides $m^2 - m$ or 3 divides $m^2 - m - 2$.

Proof by Cases:

Case 1: $m = 3q$

Case 2: $m = 3q+1$

Case 3: $m = 3q+2$

$\Bigg\}$ → by the Division Algorithm, one of these must be true for some $q \in \mathbb{Z}$.

Case 1: ~~~~ Suppose $m = 3q$. Then

$$m^2 - m = (3q)^2 - 3q = 9q^2 - 3q = 3(3q - q)$$

$\Rightarrow$ ~~~~ 3 divides $m^2 - m$ by definition of divides.

Case 2: $m^2 - m = (3q+1)^2 - (3q+1) = 9q^2 + 6q + 1 - (3q+1)$

$$= 9q^2 + 3q$$

$$\Rightarrow \quad m^2 - m = 3(3q^2 + q)$$

$\Rightarrow$ 3 divides $m^2 - m$ by def. of divides.

Case 3: Suppose $m = 3q+2$. Then

$$m^2 - m = (3q+2)^2 - (3q+2) = 9q^2 + 12q + 4 - 3q - 2$$

$$\Rightarrow \quad m^2 - m = 9q^2 + 9q + 2$$

$$\Rightarrow \quad m^2 - m - 2 = 9q^2 + 9q$$

$$\Rightarrow \quad m^2 - m - 2 = 3(3q^2 + 3q)$$

$\Rightarrow$ 3 divides $m^2 - m - 2$ by def. $\blacksquare$

(9) (5 points) Use the *definitions* of rational and/or irrational to prove the following:

For all nonzero real numbers x and y, if x is rational and y is irrational, then $\frac{x}{y}$ is irrational.

(8) (5 points) Prove that if m is an integer, then 3 divides $m^3 - m$.

Case 1: $m = 3q$
Case 2: $m = 3q+1$
Case 3: $m = 3q+2$

By division algorithm, one of these cases must be true.

Proof by cases:

- Case 1: if $m = 3q$, $m^3 - m = (3q)^3 - (3q)$

$$= 27q^3 - 3q$$

$$m^3 - m = 3(9q^3 - q)$$

so 3 divides $m^3 - m$ by def.

- Case 2: if $m = 3q+1$,

$$m^3 - m = (3q+1)^3 - (3q+1)$$

$$= 27q^3 + 27q^2 + 9q + \cancel{1} - 3q - \cancel{1}$$

$$m^3 - m = 3(9q^3 + 9q^2 + 3q - q)$$

$\Rightarrow$ 3 divides $m^3 - m$ by def.

$(3q+1)^2(3q+1)$

$(9q^2+6q+1)(3q+1)$

$27q^3 + 27q^2 + 9q + 1$

- Case 3: if $m = 3q+2$

$$m^3 - m = 27q^3 + 36q^2 + 24q + 8 - (3q+2)$$

$$= 27q^3 + 36q^2 + 21q + 6$$

$$= 3(9q^3 + 36q + 7q + 2)$$

$\Rightarrow 3 \mid (m^3 - m)$ by def.

$(9q^2 + 6q + 4)(3q+2)$

(9) (5 points) Use the *definitions* of rational and/or irrational to prove the following: For all nonzero real numbers x and y, if x is rational and y is irrational, then $\frac{x}{y}$ is irrational.

No clear

~~Cont~~ Proof by contradiction. Suppose $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$, and $\frac{x}{y}$ is rational $\mathbb{Q}$.

Then $\frac{x}{y} = \frac{p}{q}$ for some integers p, q, $q \neq 0$.

So $\frac{x}{y} = \frac{p}{q}$

$\Rightarrow xq = py$ (cross multiplying)

$\Rightarrow \frac{xq}{p} = y$

~~But x, q, and p are~~ But x is rational, so $x = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0 \Rightarrow$

$$\frac{\left(\frac{m}{n}\right) q}{p} = y$$

$$\Rightarrow \frac{mq}{np} = y$$

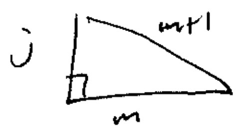
if $p \neq 0$, then y is rational since $m, q, n, p \in \mathbb{Z}$, $n \neq 0$

If $p = 0$, then $\frac{x}{y} = \frac{p}{q} = 0 \Rightarrow x = 0$, but we know x is nonzero.

$\boxed{2}$

(10) (5 points) Prove the following statement. You will need to use the Pythagorean theorem for right triangles.

Suppose that j and m are positive integers, j and m are the lengths of the legs of a right triangle, and $m+1$ is the length of the hypotenuse of the triangle. Prove that j must be an odd integer.



$\Rightarrow j^2 + m^2 = (m+1)^2$

$\Rightarrow j^2 + \cancel{m^2} = \cancel{m^2} + 2m + 1$

$\Rightarrow j^2 = 2m+1$ so j^2 is odd by def.

j^2 odd $\Rightarrow j$ is odd by previously proved theorem. $\blacksquare$

Extra Credit(2 points) Suppose $P(x)$ and $Q(x)$ are open sentences. Is

$$\Big((\exists x \in \mathbb{R})\big(P(x) \leftrightarrow Q(x)\big)\Big) \equiv \Big((\exists x \in \mathbb{R})\big(P(x)\big) \leftrightarrow (\exists x \in \mathbb{R})\big(Q(x)\big)\Big)?$$

Explain your answer. No! Suppose $P(x)$ is $x \geq 0$ and $Q(x)$ is $x < 0$. Then the LHS is false, b/c there is no x that is both ≥ 0 and < 0 at the same time. But the RHS is true because each part is true.