4:11
4:19
8 ⇒ 25 min.

# Quiz 3, Math Thought

Dr. Graham-Squire, Spring 2016

**Name:** Key

1. (5 points) Choose one of the following to prove using induction:

   (a) $2+5+8+\cdots+(3n-1)=\dfrac{n(3n+1)}{2}$

   (b) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

(a) Base step: $n=1$: $\quad 2 \stackrel{?}{=} \dfrac{1(3(1)+1)}{2} = \dfrac{4}{2}$ ✓

Inductive step: Suppose $2+5+8+\cdots+(3k-1) = \dfrac{k(3k+1)}{2}$

Want to prove $2+5+8+\cdots+(3k-1)+(3(k+1)-1) = \dfrac{(k+1)(3(k+1)+1)}{2}$

$$2+5+8+\cdots+(3k-1)+(3k+2) = \frac{k(3k+1)}{2} + 3k+2$$

$$= \frac{3k^2+k}{2} + \frac{6k+4}{2}$$

$$= \frac{3k^2+7k+4}{2} = \frac{(3k+4)(k+1)}{2}$$

$$= \frac{(k+1)(3(k+1)+1)}{2}$$ ✓

(b) Base step: $n=1$

$$\frac{1}{1\cdot 2} \stackrel{?}{=} \frac{1}{1+1} = \frac{1}{2}$$ ✓

Inductive step: Suppose $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{k(k+1)} = \dfrac{k}{k+1}$

Prove that $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)(k+2)} = \dfrac{k+1}{k+2}$

Proof: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)(k+2)} = \dfrac{k}{k+1}+\dfrac{1}{(k+1)(k+2)}$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)} = \frac{k+1}{k+2}$$ ✓

2. (2 points) Evaluation of proof: The following induction proof has at least one mistake. Explain what it is (or they are). You do not need to check the algebra, it is definitely correct–you should be looking for mistakes in the structure of the proof:

Proof that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$, by induction:

Need base step

Need to prove that $P(k+1)$ is true, which is the statement

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

Using the induction hypothesis on the left-hand side and simplifying the right-hand side we have:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Assume $P(k+1)$

$$\Rightarrow 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{(k^2+3k+2)(k+3)}{3} - (k+1)(k+2)$$

$$\Rightarrow 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k^3+6k^2+11k+6}{3} - \frac{3(k^2+3k+2)}{3}$$

$$\Rightarrow 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k^3+3k^2+2k}{3}$$

$$\Rightarrow 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

which is $P(k)$ so this verifies the proof.

First issue: Never prove a base step.

Second issue: Assumed $P(k+1)$ and proved $P(k)$, should assume $P(k)$ and prove $P(k+1)$!

3. (3 points) Consider the sequence $a_1, a_2, \ldots, a_n, \ldots$ determined recursively as follows: $a_1 = 1$, $a_2 = 1$, and for each $n$ we have the recursive definition $a_{n+2} = a_{n+1} + 3a_n$.

(a) Calculate $a_1, a_2, \ldots, a_7$.

(b) Make a conjecture about which of the entries in the sequence are even numbers.

(c) Set up an induction proof for your conjecture. You do NOT have to actually prove it, just set up and explain what you would need to do to prove it.

(a) $a_1 = 1$ ✓
$a_2 = 1$
$a_3 = 1 + 3(1) = 4$
$a_4 = 4 + 3(1) = 7$
$a_5 = 7 + 3(4) = 19$
$a_6 = 19 + 3(7) = 40$
$a_7 = 40 + 3(19) = 97$

✓ (b) For all $n$ such that 3 divides $n$, $a_n$ is even.

✓ (c) Proof: Do $n=3$ as base step ($a_3 = 4$, which is even) ✓
Inductive step: Assume that $a_k$ is even, and prove that $a_{k+3}$ is even.

0.5 for recognizing Base step, Inductive

+0.5 more for correct statements.