RESEARCH STATEMENT

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1. Introduction

My primary fields of interest are ordinary differential and difference operators, however my approach is via the formal theory and thus algebraic in nature. Specifically, I am interested in local integral transforms which relate vector spaces with difference and differential operators. Currently I am finishing a paper [Gr] calculating the local Fourier transform of connections on the punctured formal disk. Explicit formulas have already been proved in [Fa] and [Sa], and my work corroborates their results using a different method described in [Ar]. For my thesis I will apply the same techniques to develop the theoretical framework for the local Mellin transform on the punctured formal disk, as well as expressly calculating how the local Mellin transform acts. The Mellin transform is particularly interesting because it transforms a vector space with a differential operator to a vector space with a difference operator. Although much recent work has been done on the local Fourier transform, the Mellin transform has not been analyzed in the same manner.

2. Current Research: Local Fourier Transform

Definitions and Background. Consider the field of formal Laurent series $K = \mathbb{C}((z))$ and let V be a finite-dimensional vector space over K. A *connection* on V is a $\mathbb{C}$-linear operator $\nabla : V \to V$ satisfying the Leibniz identity:

$$\nabla(fv) = f\nabla(v) + \frac{df}{dz}v$$

for all $f \in K$ and $v \in V$. A choice of basis in V gives an isomorphism $V \simeq K^n$; we can then write $\nabla = \nabla_z$ as $\frac{d}{dz} + A$, where $A = A(z) \in \mathfrak{gl}_n(K)$ is the *matrix* of ∇ with respect to this basis.

We write $\mathcal{C}$ for the category of vector spaces with connections over K. Its objects are pairs (V, ∇), where V and ∇ are as defined above. Morphisms between (V_1, ∇_1) and (V_2, ∇_2) are K-linear maps $\phi : V_1 \to V_2$ such that $\phi\nabla_1 = \nabla_2\phi$. Within the category $\mathcal{C}$ there is one particular type of object that is useful to distinguish, which we describe below. Let q be a positive integer and consider the field $K_q = \mathbb{C}((z^{1/q}))$. For every $f \in K_q$, we define an *irreducible* object $E_f \in \mathcal{C}$ by

$$E_f = \left(K_q, \frac{d}{dz} + z^{-1}f\right).$$

For any connection ∇, a basis for V over a finite extension of K exists such that ∇ can be written in a *canonical form*. Such classification of differential operators goes back to [Tu] and [Le]; more modern approaches can be found in [VS2], [BV], [BBE] and [Ma]. The canonical form allows us to represent any object in $\mathcal{C}$ as a direct sum of indecomposable objects, and each indecomposable object as a successive extension of an irreducible object E_f. The canonical form is useful because calculation of the local Fourier transform reduces to just looking at irreducible objects.

Definition of local Fourier transforms. The local Fourier transforms $\mathcal{F}^{(0,\infty)}$, $\mathcal{F}^{(\infty,0)}$ and $\mathcal{F}^{(\infty,\infty)}$ for connections on formal punctured disks were introduced in [BE], and can also be viewed through the lens of microlocalization as in [Ga]. Their work on the local Fourier transform was corroborated in [Ar].

We use the following notation when we wish to stress a choice of coordinate for the category $\mathcal{C}$. $\mathcal{C}_0$ indicates the coordinate z at the point zero, and $\mathcal{C}_\infty$ indicates the coordinate $\zeta = \frac{1}{z}$ at the point at infinity. Note that $\mathcal{C}_0$ and $\mathcal{C}_\infty$ are both isomorphic to $\mathcal{C}$, but not canonically.

We let the Fourier transform coordinate of z be $\hat{z}$, with $\hat{\zeta} = \frac{1}{\hat{z}}$. Let $E = (V, \nabla_z) \in \mathcal{C}_0$ such that ∇_z has no horizontal sections and thus is invertible. The following is a precise definition for $\mathcal{F}^{(0,\infty)}E$, the other local Fourier transforms can be defined analogously. Consider on V the $\mathbb{C}$-linear operators

$$\hat{\zeta} = -\nabla_z^{-1} : V \to V \text{ and } \hat{\nabla}_{\hat{\zeta}} = -\hat{\zeta}^{-2} z : V \to V.$$

As in [Ar], $\hat{\zeta}$ extends to define an action of $\hat{K} = \mathbb{C}((\hat{\zeta}))$ on V and $\dim_{\hat{K}} V < \infty$. Then the $\hat{K}$-vector space V with connection $\nabla_{\hat{\zeta}}$ is denoted by

$$\mathcal{F}^{(0,\infty)}(E) \in \mathcal{C}_\infty,$$

which defines the functor $\mathcal{F}^{(0,\infty)} : \mathcal{C}_0 \to \mathcal{C}_\infty$. If one considers only certain subcategories of $\mathcal{C}_0$ and $\mathcal{C}_\infty$, the functors $\mathcal{F}^{(0,\infty)}$ and $\mathcal{F}^{(\infty,0)}$ will define an equivalence of categories (cf. [BE], [Ar]).

Previous results. The work of [BE] and [Ga] defined the local Fourier transform and proved some results for it, but did not expressly show how one would calculate it. In [Ar], another framework was given for the local Fourier transform, as well as explicit calculation of the Katz-Radon transform. Explicit formulas for the local Fourier transforms are proved in [Fa] and [Sa], but their methods are different: the proof in [Sa] is more geometric whereas in [Fa] algebraic techniques are used.

Our work. Our paper [Gr] corroborates the results of [Fa] and [Sa], however we apply a different method of proof. Our approach is similar to [Fa] but more straightforward. Specifically, one must ascertain the canonical form of the local Fourier transform of a given connection in order to calculate the local Fourier transform. This amounts to constructing an isomorphism between two connections, which in [Fa] is done by writing matrices of the connections with respect to certain bases. We work with operators directly, using the technique in [Ar] to define fractional powers for operators. Our method of proof is particularly useful because it generalizes to other integral transforms such as the Mellin transform.

An explicit formula for calculating $\mathcal{F}^{(0,\infty)}$ of an irreducible object is given in the following theorem, and similar theorems hold for $\mathcal{F}^{(\infty,0)}$ and $\mathcal{F}^{(\infty,\infty)}$. Let s be a nonnegative integer and r a positive integer.

Theorem 2.1. *Let $f \in K_r(z)$ with $ord(f) = -s/r$ and $f \neq 0$. Then*

$$\mathcal{F}^{(0,\infty)} E_f \simeq E_g,$$

where $g \in K_{r+s}(\hat{\zeta})$ is determined by the following system of equations:

$$f = -z\hat{z} \tag{1}$$

$$g = f + \frac{s}{2(r+s)} \tag{2}$$

3. Current Research: Local Mellin Transform

Overview. For my thesis I will prove results for the local Mellin transform that are analogous to the results proved for the local Fourier transforms. The local Mellin transform is different from the Fourier situation, however, in that it inputs a vector space with connection and outputs a vector space with a difference operator. Vector spaces with difference operators are defined in a similar fashion to vector spaces with connection, and have many related properties [VS1]. The *global* Mellin transform is discussed in [La], and my dissertation will construct the theoretical framework for the local Mellin transform as well as prove explicit formulas for its calculation. The usefulness of the Fourier transform is well-known, but similar integral transforms such as the Mellin transform have not been studied as extensively. My thesis will shed light in that direction, as well as create a link between singularities of difference and differential operators that could potentially reduce questions about difference operators to that of more-extensively studied connections.

Difference Operators on V. Let V be a finite-dimensional vector space over $K = \mathbb{C}((\theta))$. A *difference operator* on V is a $\mathbb{C}$-linear operator $\Phi : V \to V$ satisfying

$$\Phi(fv) = \varphi(f)\Phi(v)$$

for all $f \in K$, $v \in V$, where $\varphi : V \to V$ is the $\mathbb{C}$-automorphism defined below. After a choice of basis in V we can write Φ as $A\varphi$. Here $A = A(\theta) \in \mathfrak{gl}_n(K)$ is the *matrix* of Φ with respect to this basis and for $v(\theta) \in K^n$ we have

$$\varphi(v(\theta)) = v\left(\frac{\theta}{1+\theta}\right) = v\left(\sum_{i=1}^{\infty}(-1)^{i+1}\theta^i\right).$$

Remark. For all $q \in \mathbb{N}$, over the extension $K_q = \mathbb{C}((\theta^{1/q}))$, φ naturally extends to a $\mathbb{C}$-automorphism of K_q^n.

We write $\mathcal{N}$ for the category of vector spaces with difference operators over K; its properties are analogous to those of $\mathcal{C}$ above. A canonical form for objects in $\mathcal{N}$ exists and is analogous to the canonical form for connections (cf. [Pr], [VS1]). Most notably the irreducible components play the same role in both constructions.

Results to be proved. As in the Fourier case we will have several types of local Mellin transform, depending on the singularity. We refer to these as local Mellin transforms, and below we describe the definition of one such local Mellin transform. We let the Mellin transform coordinate of z be η, with $\theta = \frac{1}{\eta}$. Let $E = (V, \nabla) \in \mathcal{C}_0$ such that ∇ has no horizontal sections, thus $z\nabla$ is invertible. The following is a precise definition for the *local Mellin transform from zero to infinity*, denoted by $\mathcal{M}^{(0,\infty)}$. Consider on V the $\mathbb{C}$-linear operators

$$\theta = -(z\nabla)^{-1} : V \to V \text{ and } \Phi = z : V \to V. \tag{3}$$

We will show that θ extends to define an action of $\mathbb{C}((\theta))$ on V and $\dim_{\mathbb{C}((\theta))} V < \infty$. Then the $\mathbb{C}((\theta))$-vector space V with difference operator Φ is denoted by

$$\mathcal{M}^{(0,\infty)}(E) \in \mathcal{N},$$

which defines the functor $\mathcal{M}^{(0,\infty)} : \mathcal{C}_0 \to \mathcal{N}$.

In my dissertation I will introduce the local Mellin transforms and prove that they are well-defined. I will also show that there are inverse Mellin transforms from $\mathcal{N}$ to $\mathcal{C}$ and that these define equivalences for certain subcategories of $\mathcal{N}$ and $\mathcal{C}$. Lastly, I will give explicit formulas for calculation of the local Mellin transforms in the spirit of Theorem 2.1.

4. Research Plans

Short-term goals. In the near future I plan to continue my work in the construction and calculation of local integral transforms. Specifically, for q-difference operators there exists a classification and an integral transform of similar flavor to those for differential operators ([La], [BG]). I plan to apply the methods used in my thesis to define the structure and explicit formulas for local integral transforms arising in this context.

Long-term goals. Some other related topics that I plan to investigate are as follows:

- One can view the action of a difference operator as the action of $\mathbb{Z}$ on the formal disk. This can be generalized to look at the action of other groups on the formal disk, and we can ask the same questions: Does a classification exist and how do local transforms operate?
- The focus of my thesis is on the local Mellin transform, but it would be interesting to construct a fuller picture for how the local Mellin transform relates to the global Mellin transform. For the Fourier transform, such relationship is given in [BE], [Ga] and [Ar], and I hope to give results analogous to the results found in those papers.
- In the same way that a difference quotient "degenerates" into a derivative (via the limit process), one can view the degeneration of the Mellin transform into the Fourier transform. It would be interesting to look at what light, if any, this viewpoint would shed on the work I have done on the local Fourier and Mellin transforms.

My work with local integral transforms has touched other areas of mathematics that I find interesting, but have not had time to pursue. These include algebraic geometry, the analytic theory of differential and difference operators, and $\mathcal{D}$-modules.

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