Test 3, Linear Algebra

Dr. Adam Graham-Squire, Fall 2017

Name: ________________________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

____

(*signature*)

DIRECTIONS

1. Don't panic.
2. Show all of your work. A correct answer with insufficient work will lose points.
3. Read each question carefully and **make sure you answer the question that is asked**. If the question asks for an explanation, make sure you give one.
4. Clearly indicate your answer by putting a box around it.
5. Calculators are allowed on this exam, though they are not necessary.
6. Make sure you sign the pledge.
7. The first 10 questions are required, and I will drop your lowest score of the last three questions.
8. Number of questions = 12. Total Points = 55.

1. (6 points) Consider the following set of three polynomials in $\mathbb{P}_3$:

$$\{1 + t^2 + t^3, 1 - t, (1 - t)^2\}$$

Answer the following, and justify your answer:

(a) Is the set linearly independent?

(b) Does the set span $\mathbb{P}_3$?

(c) Is the set a basis for $\mathbb{P}_3$? If not, what could you do to it to make it a basis?

2. (3 points) Find the characteristic polynomial and eigenvalues for the matrix $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

   The characteristic polynomial should be written in factored form. Show your work for how you got your answers, but you can use a calculator or computer to check your work.

3. (6 points) Is the matrix $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ from problem 2 diagonalizable? If so, find D and P. If not, explain why it is not diagonalizable. Show your work for how you got your answers, but you can use a calculator or computer to check your work.

4. (3 points) A is a 6×6 matrix with 4 distinct eigenvalues. One eigenspace is two-dimensional. Is A diagonalizable? Your answer should be Yes, No, or Can't Say, and you should explain your reasoning.

5. (3 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

   (i) Let A be an $m \times n$ matrix. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then Col A is all of $\mathbb{R}^m$.

   (ii) To find the eigenvalues of A, reduce A to echelon form.

   (iii) A change of coordinates matrix is always invertible.

   (iv) A diagonalizable matrix is always invertible.

6. (4 points) Multiple Choice Questions. You do *not* need to show any work to receive full credit for these problems (although showing work can get you partial credit if your answer is wrong).

   (a) The vector space of polynomials of degree 3 or less is $\mathbb{P}_3$, it has dimension

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

   (b) The subspace $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ has dimension

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

   (c) If the column space of a 3×6 matrix A is 2-dimensional, what is the dimension of the null space of A?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6 (h) 7

   (d) If A is a 2×4 matrix, what is the largest possible rank of A?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6

   (e) If A is a 3×2 matrix, what is the smallest possible dimension of the nullspace of A?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6

7. (6 points) Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 2 \\ -9x_1 + 2x_2 + 5x_3 &= 3 \\ 4x_1 + x_2 - 6x_3 &= 9 \end{aligned} \quad \text{and} \quad \begin{aligned} 5x_1 + x_2 - 3x_3 &= -6 \\ -9x_1 + 2x_2 + 5x_3 &= -9 \\ 4x_1 + x_2 - 6x_3 &= -27 \end{aligned}$$

Suppose you know that the first system has a solution. Use this fact to explain why the second system also has a solution without making any row operations.

8. (6 points) $M_{2\times 3}$ is the vector space of all 2×3 matrices with normal addition and scalar multiplication. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & 0 & f \end{bmatrix}$, where a, b, f are real numbers, is a subspace of $M_{2\times 3}$.

9. (6 points) Let $A = \begin{bmatrix} 1 & -2 & 3 & 5 & 8 \\ 2 & -4 & 6 & 15 & 21 \\ 3 & -6 & 9 & 15 & 22 \\ -1 & 2 & -3 & 0 & -1 \end{bmatrix}$. An echelon form for A is $\begin{bmatrix} 1 & -2 & 3 & 5 & 8 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and its reduced echelon form is $\begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer the following (Show work if necessary):

(a) Find the dimension of and a basis for Nul A.

(b) Find the dimension of and a basis for Col A.

(c) Find the dimension of and a basis for Row A.

(d) Find the rank of A.

You only need to do two of these last three questions, but you can do all of them and I will take your 2 highest scores.

10. (6 points) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [Hint: Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

11. (6 points) Let C be the vector space of all continuous functions on the interval $[0, 1]$. Define $T : C \to C$ to be the transformation as follows: for the function f, let $T(f)$ be the *antiderivative* F of f such that $F(0) = 0$ (so, for example, $T(x^2) = \frac{x^3}{3}$).

(a) Show that T is a linear transformation

(b) Describe the kernel of T (that is, describe all functions such that $T(f) = \mathbf{0}$).

12. (6 points) Let $T : V \to W$ be a linear transformation. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, since the range of a linear transformation is always a subspace. Prove that $\dim T(H) \leq \dim H$. [Hint: Every vector in $T(H)$ has the form $T(\mathbf{x})$ for some $\mathbf{x}$ in H. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V, and write $\mathbf{x}$ as a linear combination of the $\mathbf{v}_i$. Apply T to both sides and use this to argue that a basis for $T(H)$ can have at most n vectors.]

Extra Credit (2 points): Let S be the set of all 2×2 matrices A such that A has an eigenvalue of zero. Is S a subspace of $M_{2\times 2}$?