MTH 2310, LINEAR ALGEBRA

MINITEST 4 REVIEW, DR. ADAM GRAHAM-SQUIRE

(1) True/False: If True, justify your answer with a brief explanation. If False, give a counterexample or a brief explanation.

(a) $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$.

True, this is the same as $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ which is a property of the dot product.

(b) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

True, because $\dfrac{\mathbf{y} \cdot (c\mathbf{v})}{(c\mathbf{v}) \cdot (c\mathbf{v})}(c\mathbf{v}) = \dfrac{c^2(\mathbf{y} \cdot \mathbf{v})}{c^2(\mathbf{v} \cdot \mathbf{v})}\mathbf{v} = proj_{\mathbf{v}}\mathbf{y}$.

(c) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{v}$ onto $c\mathbf{y}$. **False**, because the projection onto $\mathbf{v}$ is in the direction of $\mathbf{v}$, and the projection onto $\mathbf{y}$ is in the direction of $\mathbf{y}$. If those two vectors are not pointed in the same direction (and generally they will not), you will not get the same result for the two orthogonal projections. You could also just take any two vectors and do $\text{proj}_{\mathbf{v}}\mathbf{y}$ and vice versa to show that they will not be equal.

(d) If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^{\perp}$, then $\mathbf{v}$ must be the zero vector. **True**, because if $\mathbf{v}$ is in both, then it must be orthogonal to itself, that is $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2 = 0$, which is only true if you are the zero vector

(e) The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A that is closest to $\mathbf{b}$. **True**, that is the whole point of the least-squares solution. We derived the calculation of the least-squares solution from the orthogonal projection of $\mathbf{b}$ onto the column space of A, so they are in fact the same thing.

(2) Let $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. (a) Calculate $\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$.

Ans: $\begin{bmatrix} 3/7 \\ -1/7 \\ -5/7 \end{bmatrix}$.

(b) Find a unit vector orthogonal to $\mathbf{x}$ (Hint: first find an orthogonal vector by inspection and then normalize it).

Ans: One answer is $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix}$, but there are an infinite number of possible answers.

(3) Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every vector in Span$\{\mathbf{u}, \mathbf{v}\}$. (Hint: An arbitrary $\mathbf{w}$ in Span$\{\mathbf{u}, \mathbf{v}\}$ is of the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$.)

Ans: $\mathbf{y} \cdot \mathbf{w} = \mathbf{y} \cdot (c_1\mathbf{u} + c_2\mathbf{v}) = c_1(\mathbf{y} \cdot \mathbf{u}) + c_2(\mathbf{y} \cdot \mathbf{v}) = \mathbf{0}$ because $\mathbf{y} \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{v} = 0$.

(4) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

Ans: $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is a scalar multiple of $\mathbf{u}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$. Calculation gives

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

and

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

(5) Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

Ans: The orthogonal vectors must be linearly independent (we proved in class that orthogonal vectors are always linearly independent), and thus they will form a basis for the subspace W since we know that they span (alternatively, since there are n linearly independent vectors, and we are in $\mathbb{R}^n$, they must be a basis by the basis theorem). Since there are n of them, W has dimension n and therefore must be all of $\mathbb{R}^n$.

(6) Let W be a subspace of $\mathbb{R}^n$. Prove that dim W + dim $W^\perp = n$ by doing the following:

(a) Suppose W has an orthogonal basis $\{w_1, w_2, \ldots, w_p\}$, and let $\{v_1, v_2, \ldots, v_q\}$ be an orthogonal basis for $W^\perp$. Explain why $\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$ is an orthogonal set.

Ans: By the definition of $W^\perp$, every vector in $W^\perp$ will be orthogonal to every vector in W. In particular, this means that every one of the basis vectors in $W^\perp$ will be orthogonal to every one of the basis vectors in W. Since the original two bases were already orthogonal, the whole set put together will be orthogonal.

(b) Explain why $\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$ spans $\mathbb{R}^n$.

Ans: Let $\mathbf{a}$ be any vector in $\mathbb{R}^n$, and $\hat{\mathbf{a}}$ the orthogonal projection of $\mathbf{a}$ onto W. Then by orthogonal decomposition, $\mathbf{a} = \hat{\mathbf{a}} + \mathbf{z}$, where $\mathbf{z}$ is orthogonal to $\hat{\mathbf{a}}$. $\hat{\mathbf{a}}$ is in the column space of W because it is a projection vector onto that space, and $\mathbf{z}$ is in the column space of $W^\perp$ because $\mathbf{z}$ is orthogonal to something in W. This means that $\hat{\mathbf{a}}$ can be written as a linear combination of $\{w_1, w_2, \ldots, w_p\}$ and $\mathbf{z}$ can be written as a linear combination of $\{v_1, v_2, \ldots, v_q\}$, so $\mathbf{a}$ can be written as a linear combination of $\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$. Since $\mathbf{a}$ was an arbitrary vector in $\mathbb{R}^n$, this means that $\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$ must span $\mathbb{R}^n$, as desired. I am pretty sure there is any easier way to explain this as well, but I can't come up with it right now.

(c) Explain why this proves that dim W + dim $W^{\perp} = n$.

Ans: The dimension of span$\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\} = p+q$ because $\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$ has $p+q$ vectors in it. We also know that $\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$ is linearly independent and spans $\mathbb{R}^n$, so it must be a basis for $\mathbb{R}^n$ and thus span$\{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$ must equal n, so $p+q=n$. But dim $W = p$ and dim $W^{\perp} = q$ since that is how many basis vectors they each have, so by substitution we have dim W + dim $W^{\perp} = n$.

(7) Find a least-squares solution for $A\mathbf{x} = \mathbf{b}$ <u>and</u> compute the least-squares error for

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Ans: $A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$, $A^T\mathbf{b} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$, so the least-squares solution of $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (which you can get by either augmenting and row reducing or multiplying by the inverse of A^TA). This means that $\hat{\mathbf{b}} = A\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

and the least-squares error is $||\mathbf{b} - \hat{\mathbf{b}}|| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$.

And I just realized that this is Exercise 3 in your section 6.5 notes, so...oops.