Minitest 4, Linear Algebra

Dr. Adam Graham-Squire, Fall 2017

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

1. Show all of your work. A correct answer with insufficient work will lose points.
2. Read each question carefully, and make sure you answer the the question that is asked. If the question asks for an explanation, make sure you give one.
3. Clearly indicate your answer by putting a box around it.
4. Calculators/computers are allowed on the first question of the exam, and no calculator/computer on the other questions. You can come back to the first question on the test if you want to, but once the No Calculator portion is turned in you cannot go back to it.
5. Make sure you sign the pledge.
6. Number of questions = 5. Total Points = 30. Every question is required.

You will need some kind of computing device for this question, either one of the online calculators linked from our blackboard page, or a graphing calculator, or some other such thing (Maple, Matlab, etc). You do not need to write out every matrix/vector you find along the way (though that may be useful), but you should explain what you are doing to get your answers.

1. (8 points) (a) Find a least-squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ for the A and $\mathbf{b}$ given below.

   (b) Compute the associated least-squares error $||A\hat{\mathbf{x}} - \mathbf{b}||$ for your result from part (a).

   ~~(c) The vector $\mathbf{a} = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 4 \\ 3 \end{bmatrix}$ is in the column space of A. Calculate the distance $||\mathbf{a} - \mathbf{b}||$ from $\mathbf{a}$ to $\mathbf{b}$. (Note: your answer for (c) should be larger than your answer for (b))~~

Can round to nearest 0.000

$$A = \begin{bmatrix} -6 & -3 & 6 \\ -1 & 2 & 1 \\ 3 & 6 & 3 \\ 6 & -3 & 6 \\ 2 & -1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) $A^TA = \begin{bmatrix} 86 & 14 & 12 \\ 14 & 59 & -18 \\ 12 & -18 & 86 \end{bmatrix}$ ✓✓ $A^Tb = \begin{bmatrix} 4 \\ 1 \\ 18 \end{bmatrix}$ ✓✓

Solution to $A^TAx = A^Tb$

$$\left[\begin{array}{ccc|c} 86 & 14 & 12 & 4 \\ 14 & 59 & -18 & 1 \\ 12 & -18 & 86 & 18 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/1257 \\ 0 & 1 & 0 & 541/6285 \\ 0 & 0 & 1 & 476/2095 \end{array}\right]$$

$$\Rightarrow \boxed{\hat{x} = \begin{bmatrix} 1/1257 \\ 541/6285 \\ 476/2095 \end{bmatrix}} \checkmark$$

$$A\hat{x} = \begin{bmatrix} 1.1001 \\ 0.3986 \\ 1.20060 \\ 1.10970 \\ 0.3699 \end{bmatrix}$$

(b) $||A\hat{x} - b|| = \sqrt{(0.1001)^2 + (1-0.3986)^2 + (.2006)^2 + (0.1097)^2 + (1-0.3699)^2}$

$= \boxed{0.9061}$ ✓

Give

You can use this extra sheet to answer question 1.

No Calculator/Computer portion. Name: Key

2. (2 points) For each question, state if it is True or False. If True, briefly justify why it is true. If False, give an explanation or a counterexample to show why it is false.

(a) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

False! They will still be orthogonal, they just have different length.

(b) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

True! $\underbrace{y}_{\text{in } W} = y + \vec{0}$ orthogonal to everything.

(c) If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$.

False! $A^TA\hat{x} = A^Tb$, but we don't know if A^TA is invertible!

3. (5 points) Consider the set $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(a) Is it an orthogonal set? Explain/show your work.

$u \cdot v = -2 + 2 = 0$

$v \cdot w = -2 + 2 = 0$

$u \cdot w = -5 + 4 + 1 = 0$ ✓

$\boxed{\text{Yes}}$

3

(b) Does the set form a basis for $\mathbb{R}^3$? Justify your answer.

2 Yes! Because they are $\perp$, they must be linearly independent, and since there are 3 lin indep vectors and $\mathbb{R}^3$ has dim = 3, they must span and be a basis (by the basis Thm)

4. (5 points) Prove the following modification of the Orthogonal Decomposition theorem. According to the Orthogonal Decomposition theorem, if W is a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\mathbf{z}$ and $\hat{\mathbf{y}}$ are orthogonal and

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$

(a) Prove that $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ (That is, show that $(\mathbf{y} - \hat{\mathbf{y}}) \cdot \mathbf{u}_1 = 0$. You will need to use the fact that $\{\mathbf{u}_1, \mathbf{u}_2\}$ are orthogonal.)

(b) Explain (in words) why $\mathbf{z}$ ~~will also be~~ orthogonal to $\mathbf{u}_1$.

~~(c) Explain why (a) and (b) together imply that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.~~

$$(y - \hat{y}) \cdot u_1 = y \cdot u_1 - \hat{y} \cdot u_1$$

$$= y \cdot u_1 - \left(\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \right) \cdot u_1$$

$$= y \cdot u_1 - \frac{y \cdot u_1}{\cancel{u_1 \cdot u_1}} (\cancel{u_1 \cdot u_1}) - \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \cdot u_1$$

$u_2 \cdot u_1 = 0$ b/c orthog.

$$= y \cdot u_1 - y \cdot u_1$$

$$= 0$$

~~(b) Same process, will get~~

5. (5 points) Let $\mathbf{u}$ be a vector and W the set of all vectors $\mathbf{w}$ that are orthogonal to $\mathbf{u}$. Prove that W is a subspace.

$W = \{w \mid u \cdot w = 0\}$

- is zero in W? Yes b/c $u \cdot 0 = 0 \Rightarrow \vec{0} \perp u$.

- Addition: w_1, w_2 in W $\Rightarrow w_1 \cdot u = 0$ and $w_2 \cdot u = 0$

  is $w_1 + w_2$ in W? $u \cdot (w_1 + w_2) = u \cdot w_1 + u \cdot w_2 = 0 + 0 = 0$ ✓

- Scalar mult.: Suppose w_1 in W. Is cw_1 in W?

  $cw_1 \cdot u = c(w_1 \cdot u) = c\vec{0} = \vec{0}$ ✓

Extra Credit(up to 2 points) State if you want "2 points" or "1 point" of extra credit. If you say "1 point" then you will definitely get 1 point. If you say "2 points", then you either get 2 points or <u>none</u>. If more than 3 people say "2 points", then everyone who asks for 2 points will get zero. But if 3 or fewer say "2 points", then they all get 2 points.

2 pts	1 pt
2	13