Final Exam Review Worksheet

MTH 2310 - Linear Algebra
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(1) Find the value(s) of h that make the augmented matrix a *consistent* linear system: $\begin{bmatrix} 1 & h & 4 \\ -3 & 6 & 9 \end{bmatrix}$.

(2) A system of linear equations with more equations than unknowns is sometimes called an *overdetermined* system. Can such a system be consistent? Explain why, and give an example if it would help the explanation.

(3) Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and W be the set of all linear combinations of the columns of A.

(a) Is $\mathbf{b}$ in W? Explain why or why not.

(b) Explain without any calculation why $\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$ is in W.

(4) Construct a 4×4 matrix, *not* in echelon form, whose columns do not span $\mathbb{R}^4$. Explain why your matrix does not span.

(5) Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form for $A = \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$.

(6) Determine by inspection if the following sets are linearly dependent or linearly independent. If you cannot figure it out by inspection then find out some other way. Explain your reasoning.

(a) $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \right\}$, (b) $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -28 \\ 3 \end{bmatrix} \right\}$, (c) $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix} \right\}$,

(d) $\left\{ \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -28 \\ 3 \end{bmatrix} \right\}$, (e) $\left\{ \begin{bmatrix} -2 \\ 4 \\ 6 \\ -10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix} \right\}$

(7) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that projects each vector $\mathbf{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, so $T(\mathbf{x}) = (x_1, 0, x_3)$. Show that T is a linear transformation.

(8) Consider the linear transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$. Find a matrix A that implements the mapping $T(\mathbf{x}) = A\mathbf{x}$, and then state if the transformation is one-to-one or onto (or both or neither).

(9) Suppose a matrix B has a second column that is all zeros. What can you say about the second column of AB for any matrix A that has the correct size? Do an example to illustrate your conclusion.

(10) Find the inverses of $\begin{bmatrix} 1 & 5 \\ 5 & 11 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$, if they exist.

(11) Let H be a 5×5 matrix, and suppose the equation $H\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ has no solution. What can you say about the equation $H\mathbf{x} = \mathbf{0}$? Explain your reasoning.

(12) Calculate $\det \begin{bmatrix} 4 & 0 & 0 & 4 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{bmatrix}$.

(13) Let A and P be square matrices with P invertible. Show that $\det(PAP^{-1}) = \det A$.

(14) Let H be the following subset of $\mathbb{R}^2$ given by the set of points on or inside the unit circle. That is, $H = \{\begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1\}$. Determine if H is a subspace.

(15) Let $M_{2\times 2}$ be the vector space of 2×2 matrices with the usual addition and scalar multiplication of matrices. Let M be any 2×2 matrix. Then let H be the set of all 2×2 matrices A such that $MA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Is H a subspace of $M_{2\times 2}$? Justify your answer.

(16) Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(a) Is $\mathbf{w}$ is Col A?

(b) Is $\mathbf{w}$ in Nul A?

(17) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set that spans $\mathbb{R}^p$. Is S a basis for $\mathbb{R}^p$? Explain why or why not.

(18) Let S be the following set of vectors in $\mathbb{P}_3 : \{1+t^3, 3+t-2t^2, -t+3t^2-t^3\}$.

(a) Use coordinate vectors to find out if the vectors in S are linearly independent.

(b) Does S span $\mathbb{P}_3$?

(c) Is S a basis? If not, what would you need to add or take away from S to make it a basis?

(19) Find a basis for and state the dimension of the subspace of $\mathbb{R}^4$ of vectors of the form $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 3b + c = 0 \right\}$.

(20) Let A be a 5×6 matrix.

(a) What is the largest possible rank of A?

(b) What is the largest possible dimension for the null space of A?

(c) What is the smallest possible dimension for Row A?

(21) Diagonalize $\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$ if possible. Hint: You should be able to figure out at least one eigenvalue and corresponding eigenvector by inspection (and you could find all of them by inspection, if you are clever enough). You can also just use the regular method to find the characteristic polynomial, eigenvalues and eigenvectors.

(22) Find the characteristic polynomial and eigenvalues of $A = \begin{bmatrix} 1 & -5 \\ 5 & 2 \end{bmatrix}$.

(23) The matrix $A = \begin{bmatrix} 13 & -4 \\ 42 & -13 \end{bmatrix}$ has eigenvalues 1 and -1. Diagonalize A and use the diagonalization to calculate A^{200}.

(24) Let $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$, and let W be the set of all $\mathbf{x}$ in $\mathbb{R}^3$ orthogonal to $\mathbf{u}$.

(a) Prove that W is a subspace of $\mathbb{R}^3$.

(b) What is the dimension of W? Can you find a basis for it?

(25) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

(26) Let W be the subspace spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Write the vector $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W.

(27) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$. Is $A\mathbf{x} = \mathbf{b}$ consistent? If not, find a least-squares solution of $A\mathbf{x} = \mathbf{b}$.