ID Number: Key Test 3

1. (16 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

(i) A change of coordinates matrix is always invertible.

True. The columns are linearly independent and span because they form a basis, so it is invertible by the IMT.

(ii) If $\mathbf{u} \cdot \mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

False $u = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u \cdot v = 2 - 2 = 0$ but $u \neq 0$, $v \neq 0$

(iii) One way to find the eigenvalues of a square matrix A is to use row operations to reduce A to upper triangular form, then the diagonal entries will be your eigenvalues.

False. Row operations can change the eigenvalues.

(iv) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to -$\mathbf{v}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

True. $\text{dist}(u,v) = \|u-v\| = \sqrt{(u-v)\cdot(u-v)}$

$\text{dist}(u,-v) = \|u+v\| = \sqrt{(u+v)(u+v)}$

$= \sqrt{u \cdot u + 2u \cdot v + v \cdot v}$

only equal if $u \cdot v = 0$

2. (20 points) Multiple Choice Questions. You do not need to show any work to receive full credit for these problems (although showing work can get you partial credit if your answer is wrong).

(a) The trivial solution in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The subspace spanned by it has dimension

(a) 0 (b) 1 (c) 2 (d) 3

(b) The vector space of polynomials of degree 4 or less is $\mathbb{P}_4$, it has dimension

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5

$a+bt+ct^2+dt^3+et^4 \longleftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$ is $\mathbb{R}^5$

(c) If the null space of a 3×4 matrix is 2-dimensional, what is the dimension of the column space of A?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$\begin{array}{cc} & 4 \\ 3 & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array}$

$\dim(\mathrm{Col}\,A) + \dim(\mathrm{Nul}\,A) = n$

$? + 2 = 4 \Rightarrow ? = 2$

(d) If A is a 7×4 matrix, what is the largest possible rank of A?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6 (h) 7

$\begin{array}{cc} & 4 \\ 7 & \begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \\ 0 & \cdots & & 0 \end{bmatrix} \end{array}$

(e) If A is a 6×5 matrix, what is the smallest possible dimension of the nullspace of A?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6

$\begin{array}{cc} & 5 \\ 6 & \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \end{array}$

It could have 5 pivots and no free variables.

3. (15 points) Diagonalize the matrix $\begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}$.

$$\det \begin{bmatrix} 4-\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix} = -4 - 3\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 10$$
$$= (\lambda-5)(\lambda+2)$$

7 $\Rightarrow \boxed{D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}}$

· E. space for $\lambda = 5$ is $\begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow -x_1 = -3x_2$$

4

$$\Rightarrow x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

basis for e.space.

· E. space for $\lambda = -2$: $\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

4

$$x_1 = -\frac{1}{2} x_2$$

$$\Rightarrow x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\Rightarrow \boxed{P = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}}$

-3 for not getting basis

4. (10 points) Suppose A is a matrix such that $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Show that the only eigenvalue of A is 0. (Hint: Suppose that $A\mathbf{v} = \lambda\mathbf{v}$ for some vector $\mathbf{v}$. Now calculate $A^3\mathbf{v}$ in two different ways to conclude that $\lambda = 0$.)

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^3 v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for any v

$$A v = \lambda v \Rightarrow A^3 v = A(A(Av)) = A(A(\lambda v))$$
$$= \lambda (A(\lambda v))$$
$$= \lambda^2 (Av)$$
$$A^3 v = \lambda^3 v$$

Thus $\lambda^3 v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for any eigenvector v,

since $v \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, must have $\lambda = 0$ for

any eigenvalue.

5. (14 points) Let $A = \begin{bmatrix} 1 & -3 & 1 & -1 \\ 2 & -6 & 3 & 1 \\ -1 & 3 & -5 & -5 \end{bmatrix}$. An echelon form for A is $\begin{bmatrix} 1 & -3 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ and its reduced echelon form is $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Answer the following (Show work if necessary):

(a) Find the dimension of and a basis for Nul A.

(b) Find the dimension of and a basis for Col A.

(c) Find the dimension of and a basis for Row A.

(d) Find the rank of A.

✓✓

(a) dim (Nul A) = 1 = # of free variables

6

Basis: $x_1 - 3x_2 = 0$ $\quad x_1 = 3x_2$

$x_3 = 0$ ✓✓✓ $\Rightarrow x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$x_4 = 0$

So $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is basis for Nul A. ✓

(b) dim (Col A) = 3 = # of pivots.

3

Basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} \right\}$.

(c) dim (Row A) = dim (Col A) = 3

3

Basis is (1, -3, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)

(d) Rank(A) = dim (Col A) = 3.

2

6. (12 points) Consider the following set of four polynomials in $\mathbb{P}_2$: 10:55

$$\{2+t+2t^2, 3+2t-2t^2, 2+2t, 1+2t-t^2\}$$

15 min

Answer the following:

4 (a) Is the set linearly independent? No, more columns than rows. 11:11

(b) Does the set span $\mathbb{P}_2$? 11:16

(c) Is the set a basis for $\mathbb{P}_2$? If not, how could you modify it to make it a basis? 5 min

(b) $\begin{bmatrix} 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & -2 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & -6 & -4 & -3 \end{bmatrix}$

4 $\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 8 & 15 \end{bmatrix}$

Yes it spans.

(c) Is not a basis. To make it a basis, remove

4 the 4th polynomial (it is lin. dep. on the others according to matrix in (b)).

Basis is $\{2+t+2t^2, 3+2t-2t^2, 2+2t\}$.

8. (3 points) Let $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ -3 \\ \sqrt{2} \end{bmatrix}$. Find a unit vector in the direction of $\mathbf{v}$.

$$\|v\| = \sqrt{16+9+9+2} = \sqrt{36} = 6$$

So $u = \frac{1}{6} v = \begin{bmatrix} -2/3 \\ 1/2 \\ -1/2 \\ \sqrt{2}/6 \end{bmatrix}$ is a unit vector in the direction of v.

Extra Credit (2 points): Find by inspection an eigenvalue and its corresponding eigenvector for the matrix $\begin{bmatrix} 1 & 0 & 2 & 4 & -3 \\ 2 & 0 & 4 & -3 & 1 \\ -1 & 0 & 1 & 0 & 4 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 2 & 3 & -3 \end{bmatrix}$.

eigenvalue $= 0$

e. vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

7. (10 points) Let W= Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Show that if $\mathbf{x}$ is orthogonal to each of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, then $\mathbf{x}$ is orthogonal to every vector in W.

A vector in W is $w = c_1 v_1 + c_2 v_2 + c_3 v_3$.

$$\Rightarrow x \cdot w = x \cdot (c_1 v_1 + c_2 v_2 + c_3 v_3)$$

$$= c_1 (x \cdot v_1) + c_2 (x \cdot v_2) + c_3 (x \cdot v_3)$$

$$= c_1 (0) + c_2 (0) + c_3 (0)$$ b/c $x \perp v_1, v_2, v_3$

$$= 0$$

so $x \perp w$ for every w in W.