Test 2, Linear

Name: ________________________________

ID Number: ________________________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

1. Show all of your work. A correct answer with insufficient work will lose points.
2. Read each question carefully and make sure you answer the the question that is asked. If the question asks for an explanation, make sure you give one.
3. Clearly indicate your answer by putting a box around it.
4. Calculators are allowed on this exam.
5. Make sure you sign the pledge and write your ID on both pages.
6. The first 5 questions on the test are required, all of them will add to your final score. Of the last six questions (numbers 6 through 11), I will drop your lowest score. Thus you can choose to only do 5 of the last six questions, if you wish.
7. Number of questions = 11. Total Points = 100.

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1. (5 points) Let $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$. Find a nonzero vector in Col A and a nonzero vector in Nul A.

2. (15 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

   (i) If A and B are 2×2 matrices with columns $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{b}_1, \mathbf{b}_2$ respectively, then $AB = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 \end{bmatrix}$.

   (ii) $\det(A + B) = \det A + \det B$.

   (iii) Let A be a square matrix. If A^T is not invertible, then A is not invertible.

   (iv) If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

3. (12 points) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & -5 & 0 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -7 & 8 \end{bmatrix}$.

4. (6 points) Calculate $\dfrac{\det A}{\det B}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -3 & -1 & -7 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} k & k & k & k \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -3 & -1 & -7 & -5 \end{bmatrix}$$

and k is some nonzero real number. Hint: $\det A \neq 0$ and you do not need to actually calculate the determinants to get the answer.

5. (12 points) Calculate the determinants:

(a) $\det \begin{bmatrix} -3 & 7 & 5 & \pi & 17 \\ -2 & 1 & 8 & -3 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & -16 & 6 & -10 \\ \sqrt{3} & 97 & 2 & 4 & 6 \end{bmatrix}$

(b) $\det \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & -2 \\ 2 & -3 & 17 & 0 \\ 5 & 0 & 1 & 2 \end{bmatrix}$

Of the remaining 6 questions, your lowest score will be dropped

6. (10 points) Let A be an $m \times n$ matrix and suppose that $AD = I_m$.

   (a) Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. (Hint: Think about the equation $AD\mathbf{b} = \mathbf{b}$)

   (b) Explain why A cannot have more rows than columns.

7. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

(a) Construct a 4×2 matrix D using only 1 and 0 as entries, such that $AD = I_2$ (I_2 is the 2×2 identity matrix).

(b) Is it possible to find a 4×2 matrix C such that $CA = I_4$? Why or why not?

8. (10 points) Let H be a square matrix. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some value of $\mathbf{c}$, what can you say about the equation $H\mathbf{x} = \mathbf{0}$? Justify your answer.

9. (10 points) Suppose that A is a square matrix such that $\det(A^4) = 0$.

   (a) Explain why A cannot be invertible.

   (b) Does A have to be the zero matrix? If yes, explain why. If no, give a counterexample (that is, a 4×4 matrix A that is <u>not</u> the zero matrix but has the property that $\det(A^4) = 0$).

10. (10 points) Let $M_{2\times4}$ be the vector space of all 2×4 matrices with the usual operations of addition and scalar multiplication.

   (a) Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times4}$ such that $FA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Determine if H is a subspace of $M_{2\times4}$.

   (b) Let G be the set of all matrices in $M_{2\times4}$ with entries that are nonnegative. That is, for all $A = [a_{ij}]$ in H, we have $a_{ij} \geq 0$. Determine if G is a subspace of $M_{2\times4}$.

11. (10 points) Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. (Hint: Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}, \mathbf{w}$ in V. You will have to use the fact that T is a linear transformation, so you could start by writing out the 2 properties of linearity that are part of the definition. Now use those to show that the range is a subspace.)

Extra Credit (2 points): Suppose $A\mathbf{x} = A\mathbf{y}$ for some vectors $\mathbf{x}$ and $\mathbf{y}$. Is it true that A cannot be onto? Explain.