ID Number: Key

Test 2

1. (5 points) Let $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$. Find a nonzero vector in Col A and a nonzero vector in Nul A.

• Nonzero in Col A: $\boxed{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} = A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

• In Nul A do $\begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

$$x_1 - 7x_3 + 6x_4 = 0$$
$$x_2 + 4x_3 - 2x_4 = 0$$

$x_3 = 0 \quad x_4 = 1 \quad \Rightarrow \quad x_1 = -6$

$x_2 = 2$

So $\begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \in \text{Nul } A$.

2. (15 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

(i) If A and B are 2×2 matrices with columns $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{b}_1, \mathbf{b}_2$ respectively, then $AB = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 \end{bmatrix}$.

False, should be rows of A times columns of B.

as opposed to columns.

$a_1 b_1$ is not defined.

(ii) ~~If A is an~~ invertible $n \times n$ matrix then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *every* $\mathbf{b}$ in $\mathbb{R}^n$.

~~True, comes from Invertible Matrix Theorem, because A is onto.~~

(iii) $\det(A + B) = \det A + \det B$.

False

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \neq \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\det(0) = 0 \qquad 1 + 1 = 2$$

(iv) Let ~~A be a square matrix. If~~ A^T is not invertible, then A is not invertible.

True, comes from the ~~Invertible M.T.~~

(v) If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

False. The zero vector is where $f(t) = 0$ for all t.

3. (12 points) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & -5 & 0 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -7 & 8 \end{bmatrix}$.

This is of form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, so we just need to find

A^{-1} and B^{-1}

$$A^{-1} = \frac{1}{-5-(-6)} \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{24-21} \begin{bmatrix} 8 & 3 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 8/3 & 1 \\ 7/3 & 1 \end{bmatrix}.$$

Ans: $\begin{bmatrix} -5 & -3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 8/3 & 1 \\ 0 & 0 & 7/3 & 1 \end{bmatrix}$

4. (6 points) Calculate

$$\overset{= A}{\frac{\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -3 & -1 & -7 & -5 \end{bmatrix}}{\det\begin{bmatrix} k & k & k & k \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -3 & -1 & -7 & -5 \end{bmatrix} = B}}$$

where k is some nonzero real number. Hint: you do not need to actually calculate the determinants to get the answer.

Need to say $\det A \neq 0$.

$$\det\begin{bmatrix} k & k & k & k \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -3 & -1 & -7 & -5 \end{bmatrix} = k \cdot \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -3 & -1 & -7 & -5 \end{bmatrix}$$

So $$\frac{\det A}{k \cdot \det A} = \boxed{\frac{1}{k}}$$

5. (12 points) Calculate the determinants:

(4) (a) $\det \begin{bmatrix} -3 & 7 & 5 & \pi & 17 \\ -2 & 1 & 8 & -3 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & -16 & 6 & -10 \\ \sqrt{3} & 97 & 2 & 4 & 6 \end{bmatrix}$

Row 4 is $-2 \cdot$ Row 2

$= \det \begin{bmatrix} -3 & 7 & 5 & \pi & 17 \\ -2 & 1 & 8 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 97 & 2 & 4 & 6 \end{bmatrix} = \boxed{0}$ because has row of zeros.

(8) (b) $\det \begin{array}{c} \begin{array}{cccc} + & - & & \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & -2 \\ 2 & -3 & 17 & 0 \\ 5 & 0 & 1 & 2 \end{bmatrix} \end{array} = -(-3) \det \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -2 \\ 5 & 1 & 2 \end{bmatrix}$

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ -1 & 0 & -2 & -1 & 0 \\ 5 & 1 & 2 & 5 & 1 \end{matrix}$$

$= 0 - 20 - 3 - 0 + 2 + 4$

$= -17$

$= 3 \cdot (-17)$

$= \boxed{-51}$

Of the remaining 6 questions, your lowest score will be dropped

6. (10 points) Let A be an $m \times n$ matrix and suppose that $AD = I_m$.

   (a) Show that for any b in $\mathbb{R}^m$, the equation $A\mathrm{x} = \mathrm{b}$ has a solution. (Hint: Think about the equation $AD\mathrm{b} = \mathrm{b}$)

   (b) Explain why A cannot have more rows than columns.

(a) We know $ADb = I_m \cdot b = b$

so $A(Db) = b$

(5)

Thus, for any b, the expression $Ax = b$ has

the solution $x = Db$.

(b) By part (a), the matrix A is onto $\mathbb{R}^m$.

(5)

If A has more rows than columns $\begin{bmatrix} & \\ & \end{bmatrix}$

it could not have a pivot in every

row, thus it could not be onto.

7. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

(a) Construct a 4×2 matrix D using only 1 and 0 as entries, such that $AD = I_2$ (I_2 is the 2×2 identity matrix).

(b) Is it possible to find a 4×2 matrix C such that $CA = I_4$? Why or why not?

(a) $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer.

5

(b) No. If a matrix A has an inverse on both sides then it must be invertible and thus must be square. A is not square, so this cannot be.

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or

$$A\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0$$ so if such a C existed we would have

$$CA\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = C \cdot 0 = 0$$

But $CA = I_4 \Rightarrow CA\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$
$= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

since $0 \neq \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, this cannot be.

8. (10 points) Let H be a square matrix. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some value of $\mathbf{c}$, what can you say about the equation $H\mathbf{x} = \mathbf{0}$? Justify your answer.

If $Hx = c$ is inconsistent, H is <u>not</u> onto.

Thus by the I.M.T. the equation $Hx = 0$ will have <u>more</u> than just the trivial solution.

9. (10 points) Suppose that A is a square matrix such that $\det(A^4) = 0$.

   (a) Explain why A cannot be invertible.

   (b) Does A have to be the zero matrix? If yes, explain why. If no, give a counter example (that is, a 4×4 matrix A that is <u>not</u> the zero matrix but has the property that $A^4 = 0$).

~~(a) Suppose A^{-1} existed. Then $(A^{-1})^4$ exists, and~~

$$\cancel{(A^{-1})^4 \cdot A^4 = A^{-1} \cdot A^{-1} \cdot A^{-1} (A^{-1} \cdot A) A \cdot A \cdot A}$$

$$\cancel{= A^{-1} \cdot A^{-1} \cdot A^{-1} \cdot A \cdot A \cdot A}$$

~~etc~~

$$\cancel{= I.}$$

~~But $(A^{-1})^4 \cdot A^4 = (A^{-1})^4 \cdot 0 = 0$~~

~~and $I \neq 0$, so A^{-1} cannot exist.~~

or.

5 (a) $\det A^4 = 0 \iff (\det A)(\det A)(\det A)(\det A) = 0$

$\Rightarrow \det A = 0$

$\Rightarrow A$ is <u>not</u> invertible.

5 (b) <u>No</u>. if $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A$ then $A^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $A^4 = 0$ as well, but $A \neq 0$.

10. (10 points) Let $M_{2\times 4}$ be the vector space of all 2×4 matrices with the usual operations of addition and scalar multiplication.

(a) Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ such that $FA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Determine if H is a subspace of $M_{2\times 4}$.

(b) Let G be the set of all matrices in $M_{2\times 4}$ with entries that are nonnegative. That is, for all $A = [a_{ij}]$ in H, we have $a_{ij} \geq 0$. Determine if G is a subspace of $M_{2\times 4}$.

(a) • zero: $A = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$ is in b/c $F\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$

• addition: If $FA_1 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$ and $FA_2 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$

(5) then $F(A_1+A_2) = FA_1 + FA_2 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$

• scalar mult: If $FA = \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$ then $F(cA) = cFA$

$= c\begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$

$= \begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix}$

(b) Not a subspace! Counterexample:

(5) Let $A_1 = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}$

If G was a subspace, $-2\cdot A_1$ would be in G,

but $-2A_1$ has negative entries.

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11. (10 points) Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W (Hint: Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}, \mathbf{w}$ in V. You will have to use the fact that T is a linear transformation, so ~~make sure you state where you are using~~ properties of linearity.) you could start by writing out the two that are part of the definition.

- Zero: Since T is linear, $T(0) = 0$ so $0 \in$ range.
- addition: $T(x_1)$ and $T(x_2)$ are in range, is $T(x_1) + T(x_2) \in$ range?

  Yes, because $T(x_1) + T(x_2) = T(x_1 + x_2) \in$ range
- Scalar mult: $T(x_1) \in$ range, is $cT(x_1) \in$ range?

  Yes, because $cT(x_1) = T(cx_1) \in$ range.

Extra Credit (2 points): Suppose $A\mathbf{x} = A\mathbf{y}$ for some vectors $\mathbf{x}$ and $\mathbf{y}$. Is it true that A cannot be onto? Explain.

False. Example

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is onto and

$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

A does not have to be square!