Test 3 - MTH 1420

Dr. Graham-Squire, Spring 2012

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Name: Key

ID Number: ______________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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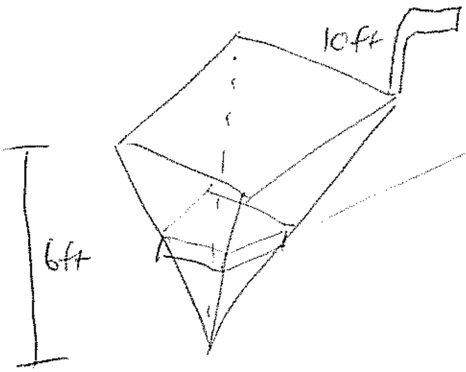
(*signature*)

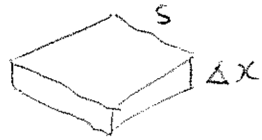
DIRECTIONS

1. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
2. Clearly indicate your answer by putting a box around it.
3. Cell phones and computers are <u>not</u> allowed on this test. Calculators <u>are</u> necessary for certain parts of the test.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge and write your ID on both pages.
6. Number of questions = 7. Total Points = 75.

ID Number: Key Test 3

1. (10 points) A pool is in the shape of an inverted pyramid with a square base. The base has sides of length 10 ft and the pyramid has a height of 6 ft. The pool is full of water and the water is pumped out of a spout that extends 2 feet above the top of the pool. Set up but **do not** ~~integrate~~ evaluate an integral that represents the amount of work needed to pump all of the water out of the pool. The density of water is 62.5 lbs/ft^3.





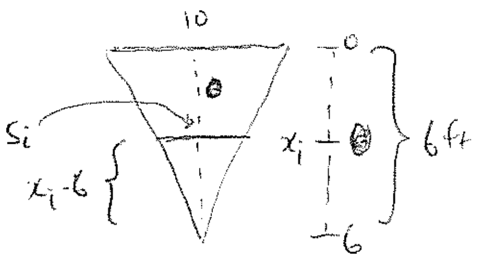
Volume of slice is $s^2 \cdot \Delta x$

$V_i = \left(\frac{5}{3}(x_i - 6)\right)^2 \cdot \Delta x$

$F_i = \frac{25}{9}(x_i - 6)^2 \, 62.5 \, \Delta x$

distance to spout

$W_i = \frac{25}{9} \cdot 62.5 \, (x_i - 6)^2 \cdot (x_i + 2) \, \Delta x$



$$\boxed{\int_0^6 \frac{25}{9}(62.5)(x-6)^2(x+2)\,dx}$$

$\frac{s_i}{x_i - 6} = \frac{10}{6}$

$s_i = \frac{\cancel{10}^{\,5}(x_i - 6)}{\cancel{6}_{\,3}}$

2. (10 points) Determine whether the sequence converges or diverges. If it converges, find the limit. Make sure to show your work!

(a) $a_n = \sqrt[n]{3^{1+2n}}$ $= \left(3^{(1+2n)}\right)^{\frac{1}{n}} = 3^{\frac{1}{n}+2}$

$$\lim_{n\to\infty} 3^{\frac{1}{n}+2} = 3^{0+2} = 3^{2} = \boxed{9}$$

(b) $a_n = \dfrac{e^{2n}+1}{e^n+e^{-n}} \cdot \dfrac{\frac{1}{e^n}}{\frac{1}{e^n}} = \dfrac{e^n + \frac{1}{e^n}}{1+\frac{1}{e^{2n}}}$

~~converge~~

$$\lim_{n\to\infty} \frac{e^n + \underbrace{\frac{1}{e^n}}_{\to 0}}{1+\underbrace{\frac{1}{e^{2n}}}_{\to 0}} = \frac{\infty + 0}{1+0} \to \cancel{\infty}$$

$\boxed{\text{diverges}}$

3. (10 points) Determine whether the series is convergent or divergent, make sure to state which test you are using. If it converges, find the exact sum. If you cannot find the exact sum, use a remainder estimate to find the sum to the nearest 0.01.

(a) $\sum_{n=1}^{\infty} 2^{n+1} \cdot 10^{-2n+3}$ $= \sum \frac{2^{n+1}}{10^{2n}} \cdot 10^3$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{100}\right)^n \cdot 2000$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{50}\right)^n \cdot 2000$$

$n=1 \Rightarrow 40 = a$

$r = \frac{1}{50}$

geometric. Converges to $\frac{40}{1-\frac{1}{50}} = \frac{40}{\left(\frac{49}{50}\right)} = \boxed{\frac{2000}{49}}$

(b) $\sum_{n=1}^{\infty} \frac{(-4)^n \cdot 7}{3^n(n+2)}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{(-4)^{n+1} \cdot \cancel{7} \cdot 3^n \; (n+2)}{(-4)^n \cdot \cancel{7} \cdot 3^{n+1} \; (n+3)} \right|$

$$= \lim_{n\to\infty} \left| \left(\frac{4}{3}\right) \cdot \left(\frac{n+2}{n+3}\right) \right|$$

goes to 1

$$= \frac{4}{3} > 1$$

diverges by the Ratio test.

4. (10 points) Determine whether the series is convergent or divergent, make sure to state which test you are using. <u>If it converges, find the exact sum</u>. <u>If you cannot find the exact sum, use a remainder estimate to find the sum to the nearest</u> 0.01.

(a) $\sum_{n=1}^{\infty} \frac{3+\cos n}{n} \geq \sum_{n=1}^{\infty} \frac{2}{n}$ $\qquad 2 \leq 3+\cos n \leq 4$

$= 2 \sum_{n=1}^{\infty} \frac{1}{n}$

diverges (harmonic series)

$\Rightarrow$ $\boxed{\sum \frac{3+\cos n}{n} \text{ diverges by comparison}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by p-test $(p = 4 > 1)$

Find n such that $\int_n^{\infty} \frac{1}{x^4}\,dx < 0.01$

$= \lim_{b\to\infty} \int_n^b \frac{1}{x^4}\,dx$

$= \lim_{b\to\infty} -\frac{1}{3x^3}\Big|_n^b = \lim_{b\to\infty} -\frac{1}{3b^3} + \frac{1}{3n^3}$

$= \frac{1}{3n^3}$

Need $\frac{1}{3n^3} < 0.01$

$\frac{1}{0.03} < n^3$

$\sqrt[3]{33.33} < n$

$3.21 < n$

Need $n = 4$

$\Rightarrow$ approximate sum is

$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} = 1.0782$

$\approx$ 1.08

5. (10 points) Determine whether the series is convergent or divergent, make sure to state which test you are using. If it converges, state whether or not it is absolutely convergent.

(a) $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n+1}}{n}$

alternating.

Check: is $\frac{\sqrt{n+1}}{n}$ decreasing?

$\frac{\sqrt{2}}{1}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{3}, \frac{\sqrt{5}}{4}, \ldots$ yes, the power on bottom is higher

$\lim_{n\to\infty}\frac{\sqrt{n+1}}{n} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n^2}} = \lim_{n\to\infty}\sqrt{\frac{1}{n}+\frac{1}{n^2}}$

$= \sqrt{0+0}$

$= 0$

So $\underline{\text{converges}}$ by A.S.T.

but $\sum\left|(-1)^n\frac{\sqrt{n+1}}{n}\right| = \sum\frac{\sqrt{n+1}}{n}$ does $\underline{\text{not}}$ converge. Compare to

$\sum\frac{1}{n^{1/2}}$: $\lim_{n\to\infty}\frac{\left(\frac{\sqrt{n+1}}{n}\right)}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}} = 1$

and $\sum\frac{1}{n^{1/2}}$ diverges (p-test, $p=\frac{1}{2}<1$)

so $\sum(-1)^n\frac{\sqrt{n+1}}{n}$ is $\underline{\text{not}}$ absolutely convergent.

(b) $\sum_{n=1}^{\infty}\frac{1}{n+2}-\frac{1}{n+5}$

$= \left(\frac{1}{3}-\cancel{\frac{1}{6}}\right)+\left(\frac{1}{4}-\cancel{\frac{1}{7}}\right)+\left(\frac{1}{5}-\cancel{\frac{1}{8}}\right)+\left(\cancel{\frac{1}{6}}-\cancel{\frac{1}{9}}\right)+\left(\cancel{\frac{1}{7}}-\frac{1}{10}\right)+\left(\cancel{\frac{1}{8}}-\frac{1}{11}\right)+\left(\cancel{\frac{1}{9}}-\frac{1}{12}\right)+\ldots$

$\Rightarrow S_n = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}$

$\lim_{n\to\infty} S_n = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ $\Rightarrow$ $\boxed{\text{converges.}}$

and $\left|\frac{1}{n+2}-\frac{1}{n+5}\right|$ is same because none of the terms are negative $\Rightarrow$ $\boxed{\text{absolutely convergent}}$

Make extra credit.

6. (10 points) Determine whether the series is convergent or divergent, make sure to state which test you are using.

(a) $\sum_{n=1}^{\infty} \pi^n \cdot e^{-n} = \sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$

$r > 1$

$\pi \approx 3.14$

$e \approx 2.78$

$\Rightarrow \frac{\pi}{e} > 1$

$\boxed{\text{diverges}}$. Geometric with $r = \frac{\pi}{e} > 1$

(b) $\sum_{n=1}^{\infty} \frac{n^3 - 3n + 4}{3n^3 + n^2 + 1}$

$$\lim_{n\to\infty} \frac{n^3 - 3n + 4}{3n^3 + n^2 + 1} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1 - \frac{3}{n^2} + \frac{4}{n^3}}{3 + \frac{1}{n} + \frac{1}{n^3}} = \frac{1}{3} \neq 0$$

So series <u>diverges</u> by the test for divergence.

7. (15 points) Find the radius of convergence and the interval of convergence of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-5)^n}{\sqrt[3]{n+2}}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(x-5)^n} \cdot \frac{\sqrt{n+2}}{\sqrt{n+3}} \right|$ → goes to 1

$= |x-5|$

$\Rightarrow$ need $|x-5| < 1$ so $\boxed{\text{radius of conv} = 1}$

$-1 < x-5 < 1$

$4 < x < 6$

$\boxed{\Rightarrow \text{int. of convergence } [4, 6)}$

check endpoints: at $x=4$ get $\displaystyle\sum \frac{(-1)^n}{\sqrt{n+2}}$ converges by A.S.T.

at $x=6$ get $\displaystyle\sum \frac{1}{\sqrt{n+2}}$ diverges b/c

$\dfrac{1}{\sqrt{n-2}} > \dfrac{1}{\sqrt{n}}$ and $\sum \frac{1}{n^{1/2}}$ diverges (p-test w/ $p = \frac{1}{2}$)

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$\Rightarrow$ 2:7
9:57

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!\, x^n}{(n+1)8^n}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{n+1}{(n+2)} \cdot \frac{8^n}{8^{(n+1)}} \right|$

$= \displaystyle\lim_{n\to\infty} \left| (n+1) \cdot x \cdot \left(\frac{n+1}{n+2}\right) \cdot \frac{1}{8} \right|$ → goes to 1

$= \infty \cdot \dfrac{x}{8} = \infty$ for all $x \neq 0$

$\Rightarrow$ Radius of convergence $= 0$
interval of convergence $= \{0\}$

Extra Credit(2 points) For one of the series in questions (5) or (6), either find the exact sum (if possible) or use a remainder theorem to approximate the sum to within 0.5.

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{20}{60}+\frac{15}{60}+\frac{12}{60}=\boxed{\frac{47}{60}}$$

10:01