ID Number: Key MiniTest 4

1. (10 points)

(a) Use known power series to determine the power series representation for

$$p(x) = \frac{-5x}{(2+5x)^2}.$$

(b) State the radius of convergence.

(a) $$\frac{1}{2+5x} = \frac{1}{2}\left(\frac{1}{1+\frac{5}{2}x}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{5}{2}x\right)^n = \sum_{n=0}^{\infty}\frac{5^n}{2^{n+1}}x^n$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2+5x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{5^n}{2^{n+1}}x^n\right)$$

$$\Rightarrow \frac{-1}{(2+5x)^2}\cdot 5 = \sum_{n=1}^{\infty}\frac{5^n}{2^{n+1}}\cdot n x^{n-1}$$

$$\Rightarrow \frac{-5x}{(2+5x)^2} = \boxed{\sum_{n=1}^{\infty}\frac{5^n}{2^{n+1}}\cdot n x}$$

$$\Rightarrow \left|\frac{5}{2}x\right| < 1$$

$$|x| < \frac{2}{5}$$

$$\Rightarrow \boxed{R = \frac{2}{5}}$$

2. (10 points) Find the Taylor series for $f(x) = \sin x$ at $a = \pi$. (Note: Assume that f has a power series expansion. Do not show that $R_n \to 0$.)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

etc.

$$\Rightarrow f(x) = \frac{-1}{1!}(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots$$

$f(\pi) = 0$

$f'(\pi) = -1$

$f''(\pi) = 0$

$f'''(\pi) = 1$

$\vdots$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}}$$

1/2 1/2 1/2 1/2

$R = \infty$, though you don't need it.

3. (10 points) Use MacLaurin series to evaluate

$$\int x \cdot e^{-x^3} dx$$

Leave your answer as a power series.

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{-x^3} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}$$

$$\Rightarrow x e^{-x^3} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{n!}$$

$$\int x e^{-x^3} = \int$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{(3n+2)\cdot(n!)} + C}$$

4. (10 points) Solve the differential equation

$$y' - e^y = e^y \cos x.$$

Simplify your answer and write it in the form $y =$ ____________.

$$y' = e^y + e^y \cos x$$

$$y' = e^y (1 + \cos x) \quad \checkmark\checkmark\checkmark$$

$$\frac{dy}{dx} = e^y (1 + \cos x) \quad \checkmark$$

$$\int e^{-y}\, dy = \int (1 + \cos x)\, dx \quad \checkmark\checkmark$$

$$-e^{-y} = x + \sin x + C \quad \checkmark\checkmark$$

$$e^{-y} = -(x + \sin x + c) \quad \checkmark$$

$$-y = \ln\left(-(x + \sin x + c)\right) \quad \checkmark$$

$$\boxed{y = -\ln\left(-(x + \sin x + c)\right)}$$

5. (10 points) Is the family of equations

$$y = f(x) = \frac{c \ln x}{x}$$

a solution to the differential equation $x^2y' + xy = c$? Justify your answer.

$$y' = \frac{\frac{c}{x} \cdot x - 1 \cdot c \ln x}{x^2} = \frac{c(1 - \ln x)}{x^2}$$

$$\cancel{x^2}\left(\frac{c(1-\ln x)}{\cancel{x^2}}\right) + \cancel{x}\left(\frac{c \ln x}{\cancel{x}}\right) \stackrel{?}{=} c$$

$$c - \cancel{c \ln x} + \cancel{c \ln x} \stackrel{?}{=} c$$

$$c = c$$

Extra Credit(1 point) Find a solution to the differential equation

$$y''' = -8y + 5.$$

$y = e^{-2x} \Rightarrow y' = -2e^{-2x}$

$y'' = 4e^{-2x}$

$y''' = -8e^{-2x} = -8y$

$\Rightarrow$ if $y = e^{-2x} + c$

$y''' = -8e^{-2x} = -8(y-c)$

$= -8y + 8c$

Need $8c = 5 \Rightarrow c = \frac{5}{8}$

So $\boxed{y = e^{-2x} + \frac{5}{8}}$