Test 3 - MTH 2410

Dr. Graham-Squire, Fall 2013

6:36

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

DIRECTIONS

1. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
2. Clearly indicate your answer by putting a box around it.
3. Calculators are allowed on all parts of the in-class portion of the test except for the last — questions, for which no technology is allowed. Even on questions where technology is allowed, you should still show all of your work. Computers and calculators are allowed on the take home part of the test, and instructions are given on that part.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge.
6. Number of questions = 9. Total Points = 70.

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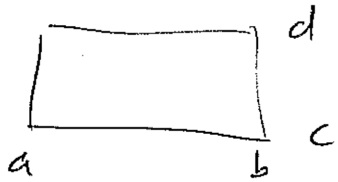
1. (6 points) TRUE OR FALSE. Circle the correct answer. If false, give a counterexample or explain (briefly) why it is false. If true, no explanation is necessary (though if you are wrong, an explanation can get you some partial credit).

(a) **True or False:** If $f(x,y)$ is integrable on the rectangle $a \le x \le b$, $c \le y \le d$, then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

True: Region is ~~squa~~ rectangular

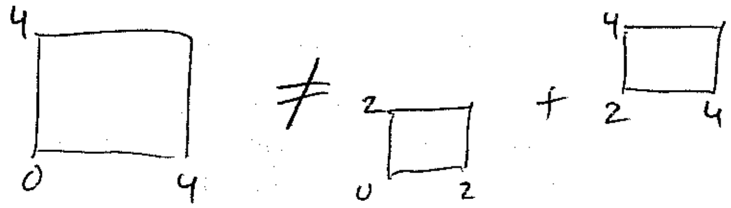
so either order of integration is fine.

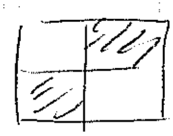


(b) **True or False:**

$$\int_0^4 \int_0^4 x^3\sqrt{\sin^2(x^2y^2)}\,dy\,dx = \int_0^2 \int_0^2 x^3\sqrt{\sin^2(x^2y^2)}\,dy\,dx + \int_2^4 \int_2^4 x^3\sqrt{\sin^2(x^2y^2)}\,dy\,dx$$

False:





(c) **True or False:** The volume of a sphere of radius R is given by

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^R \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

False: should be $\int_0^{\pi}$ for ϕ. $0 \le \phi \le \pi$.

For questions 2-8, you should use whatever method and coordinate system you deem most appropriate in order to calculate the integrals. For some integrals, it may be useful to convert to polar, cylindrical, or spherical coordinates. For some integrals it may be helpful to change the order of integration, or do a change of coordinates, in order to make the integral more manageable.

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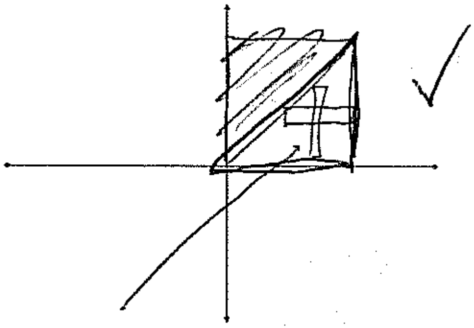
2. (8 points) Evaluate the iterated integral $\int_0^1 \int_y^1 \sin(x^2)\, dx dy$.

$0 \le y \le 1$

$y \le x \le 1$



$\int_y^1 \sin(x^2)\, dx$ is <u>not</u> integrable

switch to

$0 \le y \le x$

$0 \le x \le 1$

$$\int_0^1 \int_0^x \sin(x^2)\, dy\, dx$$

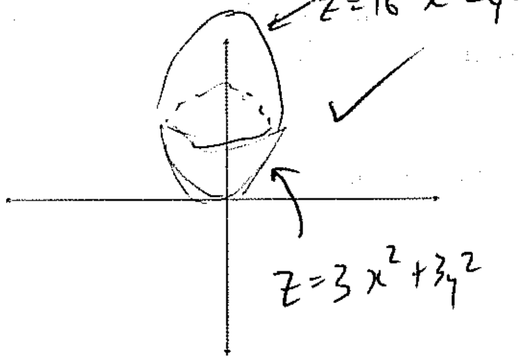
$$= \int_0^1 x \sin(x^2)\, dx$$

$$= -\frac{1}{2}\cos(x^2)\Big|_0^1$$

$$= -\frac{1}{2}\cos(1) + \frac{1}{2}\underbrace{\cos(0)}_{=1}$$

$$= \boxed{-\frac{1}{2}\cos(1) + \frac{1}{2}}$$

3. (8 points) Find the volume of the region enclosed by the paraboloids $z = 16 - x^2 - y^2$ and $z = 3x^2 + 3y^2$.



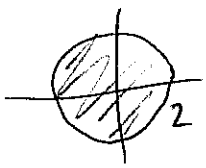
Set equal:

$$3x^2+3y^2=16-x^2-y^2$$

$$4x^2+4y^2=16$$

$$x^2+y^2=4$$

$\Rightarrow$



Do polar: $0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$V = \int_0^2 \int_0^{2\pi} \left(16-x^2-y^2-(3x^2+3y^2)\right) r \, d\theta \, dr$$

$$= 2\pi \int_0^2 (16-4r^2) r \, dr$$

$$= 2\pi \int_0^2 (16r-4r^3) \, dr$$

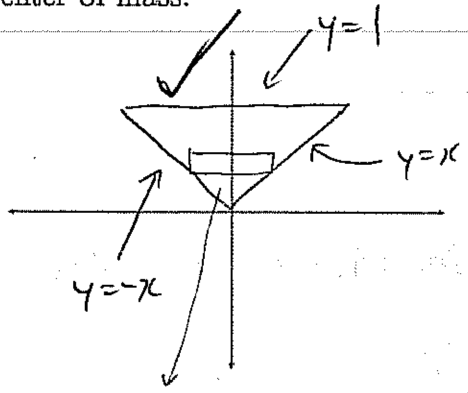
$$= 2\pi \left(8r^2-r^4 \Big|_0^2\right)$$

$$= 2\pi (32-16) = \boxed{32\pi}$$

4. (8 points) Consider the triangular lamina with corners at (0,0), (1,1) and (-1,1), with density function $\rho(x, y) = y^2$.

(a) Sketch the region of integration and determine (from the shape of the region and the density functions) approximately where the center of mass should be. Explain your reasoning.

(b) Set up, but do not integrate, the integrals that you would need in order to find the center of mass.



(a) center of mass: $\bar{x} = 0$ b/c of symmetry of shape.

$\rightarrow \bar{y} \approx 0.8$ b/c gets fatter as you go up, and density $= y^2$ increases.

$0 \le y \le 1$

$-y \le x \le y$

$$m = \int_0^1 \int_{-y}^{y} y^2 \, dx \, dy$$

$$M_x = \int_0^1 \int_{-y}^{y} y^3 \, dx \, dy$$

$= 2\int_0^1 y^3 = \frac{2}{4} y^4 \Big|_0^1 = \frac{2}{4} = \frac{1}{2}$

$2\int_0^1 y^4 \, dy$

$\frac{2}{5} y^5 \Big|_0^1$

$\frac{2}{5}$

$\dfrac{\frac{2}{5}}{\frac{1}{2}} = \frac{4}{5} = \boxed{0.8}$

(c)

dead-on! Wow!

5. (8 points) Calculate the surface area of the graph of $f(x, y) = x^2 - 3xy - y^2$ over the region inside the circle $x^2 + y^2 = 4$.

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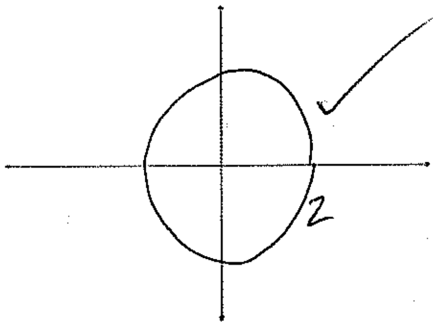

$f_x = 2x - 3y$ ✓

$f_y = -3x - 2y$

$\sqrt{1 + f_x^2 + f_y^2}$

$= \sqrt{1 + 4x^2 - \cancel{12xy} + 9y^2 + 9x^2 + \cancel{12xy} + 4y^2}$

$= \sqrt{1 + 13(x^2 + y^2)}$ ✓

Integral is $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{1 + 13(x^2+y^2)}\, dx\, dy$ ✓✓

in polar is $\displaystyle\int_0^{2\pi} \int_0^2 \sqrt{1 + 13r^2}\, r\, dr\, d\theta$ ✓✓

$= 2\pi \left(\frac{1}{26} \cdot \frac{2}{3} (1 + 13r^2)^{3/2} \right) \Big|_0^2$

$= \frac{2\pi}{39} \left(53^{3/2} - 1^{3/2} \right)$

$= \boxed{\frac{2\pi}{39} \left(53^{3/2} - 1 \right)}$ ✓

6. (8 points) Set up, but do not integrate, an integral to calculate the mass of the solid bounded above by $z = x^2 - y^2 + 4$, below by the xy-plane, within the region given by $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$, and with density function $\rho(x, y, z) = z$.

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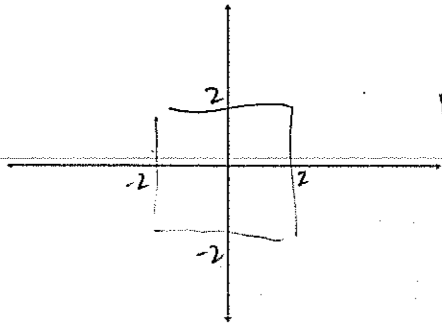

$$m = \int_{-2}^{2} \int_{-2}^{2} \int_{0}^{x^2-y^2+4} z \, dz \, dy \, dx$$

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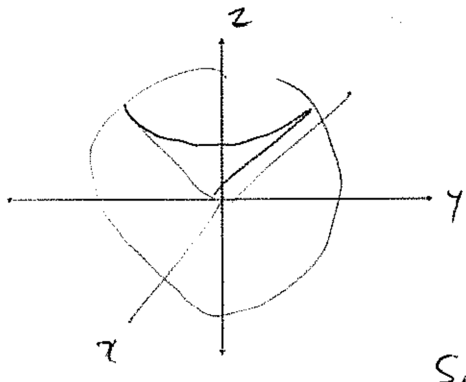
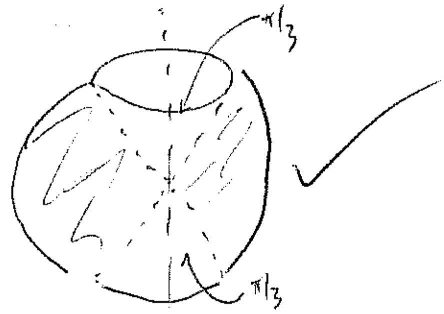
7. (8 points) Find the volume of the region inside the sphere of radius 6 but lying outside the cone given by $3z^2 = x^2 + y^2$.





sphere $\Rightarrow$ $x^2+y^2+z^2 = 36$

intersect at $\quad 3z^2 + z^2 = 36$

$4z^2 = 36$

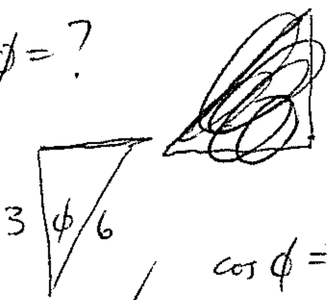
$z^2 = 9 \quad \Rightarrow z = \pm 3$

in spherical, $\quad 0 \le \rho \le 6$

$0 \le \theta \le 2\pi$

$\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$

$\phi = ?$



$\cos\phi = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \phi = 60° = \frac{\pi}{3}$

$$V = \int_0^6 \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

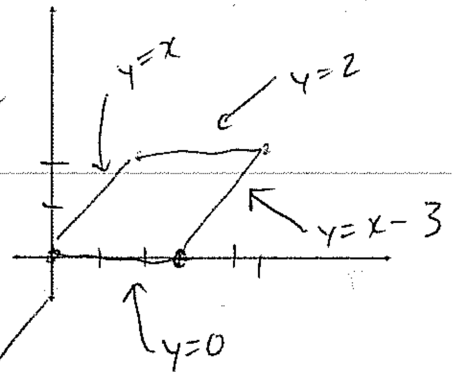
$$= 2\pi \int_0^6 \rho^2 \left(-\cos\phi \Big|_{\pi/3}^{2\pi/3}\right) d\rho$$

$$= 2\pi \int_0^6 \rho^2 \left(-\left(-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\right) d\rho$$

$$= 2\pi \quad \frac{1}{3}\rho^3 \Big|_0^6$$

$$= \frac{2\pi}{\cancel{3}} \cdot 36 \cdot \cancel{6}^{2} = \boxed{144\pi}$$

8. (8 points) Calculate $\iint\limits_R (x-y)^3 dA$, where R is the region inside the parallelogram given by connecting the points (0,0), (3,0), (5,2) and (2,2).



$x-y=0$
$x-y=3$
let $u=x-y$
$0 \le u \le 3$

Let $v=y \Rightarrow 0 \le v \le 2$

$x = u+v$
$y = v$

$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = 1 \qquad \frac{\partial y}{\partial v} = 1$

Jacobian $= |1\cdot 1 - 1\cdot 0| = 1$

$$\Rightarrow \iint\limits_R (x-y)^3 dA = \int_0^3 \int_0^2 u^3 \cdot 1 \cdot dv\, du$$

$$= \int_0^3 2u^3\, du$$

$$= \frac{1}{2}u^4 \Big|_0^3 = \boxed{\frac{81}{2}}$$

9. (8 points) Find the maximum value of the function $f(x, y) = xy$ subject to the constraint that $x^2 + 4y^2 = 16$.

$g(x,y) = x^2 + 4y^2$ at $c = 16$

$f_x = y$ $\quad f_y = x$

$\Rightarrow y = \lambda 2x \qquad x = \lambda 8y$

$g_x = 2x \quad g_x = 8y$

$\frac{y}{2x} = \lambda \Longrightarrow x = \left(\frac{y}{2x}\right) 8y$

$x^2 = 4y^2$

$x^2 + 4y^2 = 16 \Leftarrow$ constraint function.

$\Rightarrow 4y^2 + 4y^2 = 16$

$y^2 = 2$

$y = \pm\sqrt{2} \Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$

$\Rightarrow$ Max of $f(x,y) = xy$ occurs at $(2\sqrt{2}, \sqrt{2})$

$(2\sqrt{2}, -\sqrt{2})$

$(-2\sqrt{2}, \sqrt{2})$

or $(-2\sqrt{2}, -\sqrt{2})$

$f(2\sqrt{2}, \sqrt{2}) = 2\sqrt{2} \cdot \sqrt{2}$

$= \boxed{4}$

also at $(-2\sqrt{2}, -\sqrt{2})$