Calculus III, Final Exam Review Answers

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1. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

   (a) $\mathbf{u} = \langle -4, 3, -6 \rangle, \quad \mathbf{v} = \langle 16, -12, 24 \rangle$.

   (b) $\mathbf{u} = \langle -4, 3, -6 \rangle, \quad \mathbf{v} = \langle 5, 9, 1 \rangle$.

   Ans: (a) parallel, (b) neither.

2. Find a set of parametric equations for the line given by the intersection of the planes $3x - 3y - 7z = -4$ and $x - y + 2z = 3$. (Hint: find the normal vectors for each of the given planes. The cross of the normals will give you the direction vector for the line, and then you just need to find a point that lies on both of the planes.)

   Ans: x=t, y=-1+t, z=1

3. Describe and sketch the surfaces:

   (a) $y = \cos x$

   (b) $16x^2 + 16y^2 - 9z^2 = 0$

   (c) $v\mathbf{i} + 2\cos u\mathbf{j} + 2\sin u\mathbf{k}$

   Ans: (a) Cylindrical surface with lines parallel to the z-axis, and $y = \cos x$ as the graph in the xy-plane (looks like a Sun chip. Yummy).

   (b) Cone revolved around the z-axis.

   (c) Cylinder of radius 2 around the x-axis.

4. Convert the rectangular equation $x^2+y^2+z^2 = 16$ to an equation in (a) cylindrical coordinates and (b) spherical coordinates.

   Ans: (a) $r^2 + z^2 = 16$ (b) $\rho = 4$

5. Sketch and describe the space curve given by the intersection of the plane $x - y = 0$ and the surface $x^2 + z^2 = 4$. Use the parameter $x = t$ to find a vector-valued function for the space curve.

   Ans: The curve is a diagonal slice along the cylinder of radius 2 which is revolved around the y-axis. $< t, t, \sqrt{4 - t^2}$ is the upper half, $0 \le t \le 2$. The lower half is given by $< t, t, -\sqrt{4 - t^2}$

6. Evaluate the limit: $\displaystyle\lim_{t\to 0} \left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^{t}\mathbf{k} \right)$.

   Ans: $< 2, 1, 1 >$

7. For the vector-valued functions $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$ and $\mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{t}\mathbf{k}$, find

   (a) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

   (b) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$

   (c) $D_t[\mathbf{u}(t) - 2\mathbf{r}(t)]$

   Ans: (a) 0,

   (b) $(-1/t)\sin t - (1/t^2)\cos t + t\sin t - \cos t)\mathbf{i} - ((1/t)\cos t - (1/t^2)\sin t - t\cos t - \sin t)\mathbf{j} + 0\mathbf{k}$,

   (c) $-\cos t\mathbf{i} + \sin t\mathbf{j} + ((-1/t^2) - 2)\mathbf{k}$

8. A projectile is fired from ground level at an angle of $20°$ with the horizontal. The projectile has a range of 95 feet. Find the minimum initial velocity.

   Ans: 68.77 ft/sec

9. Find the curvature K of $\mathbf{r}(t) = \langle 2t, 5\cos t, 5\sin t\rangle$.

   Ans: 5/29

10. Find the limit and discuss the continuity of the function: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$.

    Ans: The limit equals zero (you can prove this using polar coordinates), and since this is the only place that would make the denominator zero, the function is continuous everywhere except at (0,0).

11. For $f(x,y) = \cos(x-2y)$, find all second partial derivatives and confirm that the mixed partials are equal.

    Ans: $f_{xy} = f_{yx} = 2\cos(x-2y)$, $f_{xx} = -\cos(x-2y)$, $f_{yy} = -4\cos(x-2y)$

12. A right circular cone is measured and the radius and height are found to be $r = 2in$ and $h = 5in$. The possible error in each measurement is $\frac{1}{8}$ inch. Use differentials to approximate the maximum possible error in the calculation of the volume ($V = \frac{1}{3}\pi r^2 h$).

    Ans: $\pm\pi\, in^3$

13. A team of oceanographers is mapping topography to assist in the recovery of a crashed helicopter. They develop the model

    $$H = 250 + 30x^2 + 50\sin\frac{\pi y}{2}, \qquad 0 \le x \le 2, 0 \le y \le 2$$

    where H is height in meters (above sea level), and x and y are distances in *kilometers*.

    (a) What is the height of the helicopter if it is located at the point $x = 1$ and $y = 0.5$?

    (b) What is the direction of greatest steepness from the location of the helicopter?

    (c) How steep is the slope if you go in the direction of greatest steepness?

    (d) How steep is the slope if you travel from the helicopter in the direction of the point (2,2)?

    Ans: (a) 280+25$\sqrt{2}$.

    (b) $\langle 60, 12.5\pi\sqrt{2}\rangle$

    (c) $\sqrt{3600 + 312.5\pi^2}$

    (d) $(120 + 37.5\pi\sqrt{2})/\sqrt{13}$

14. Find an equation for the tangent line and parametric equations of the normal line to the surface $f(x,y) = \sqrt{25-y^2}$ at the point (2,3,4).

    Ans: $3y + 4z = 25$ and $x = 2$, $y = 3t + 3$, $z = 4t + 4$.

15. Examine the surface $z = 50(x+y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$ for relative extrema and saddle points. Identify all such points, then use a computer to graph the surface and check your work.

    Ans: Relative max at (10,14, 199.4), saddle points at (10,-14, -349.4) and (-10,14, -200.6), and relative min at (-10,-14,-749.4)

16. The production function for a candy manufacturer is $f(x, y) = 4x + xy + 2y$, where x is the number of units of labor and y is the number of unit of capital. Assume that units of labor cost \$20 and units of capital cost \$4, and the total amount of money available for both labor and capital is \$2000. Write a constraint equation and then find the maximum production level for this manufacturer.

    Ans: $f(49.4, 253) = 13,201.8$

17. Evaluate the iterated integral. Change order of integration or coordinates as needed.

    (a) $\displaystyle\int_0^2 \int_{x^2}^{2x} (x^2 + 2y)\, dy\, dx$

    (b) $\displaystyle\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\, dy\, dx$

    Ans: (a) 88/15, (b) 32π

18. Find the volume of the solid:

    (a) bounded by the graphs of $z = x + y$, $z = 0$, $y = 0$, $x = 3$, and $y = x$.

    (b) bounded by the graphs of $z = 0$ and $z = 4$, outside the cylinder $x^2 + y^2 = 1$ and inside the hyperboloid $x^2 + y^2 - z^2 = 1$. You can use a computer to help visualize the region, but you should be able to do the integral by hand.

    Ans: (a) 27/2, (b) $64\pi/3$

19. Find the area of the surface $f(x) = 4 - x^2$ over the region given by the triangle bounded by the graphs of $y = x$, $y = -x$, and $y = 2$. You can use a computer to integrate the integral.

    Ans: 7.0717

20. Evaluate the integral $\displaystyle\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \frac{1}{1 + x^2 + y^2 + z^2}\, dz\, dy\, dx$. Hint: once you have set up the integral correctly, it may be helpful to rewrite the integrand as $1 - \dfrac{1}{\text{something}}$.

    Ans: $\frac{\pi}{2}(5 - \arctan 5)$

21. Find the center of mass of the solid bounded below by $x^2 + y^2 + z^2 = 25$ and above by $z = 4$. Assume that the density is constant.

    Ans: (0,0,-1/8)

22. Use the change of variables $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$ to evaluate the double integral $\displaystyle\iint_R \ln(x + y)\, dA$ where R is the square region with corners at (1,2), (2,1), (3,2) and (2,3).

    Ans: 2.751

23. Evaluate the line integral $\displaystyle\int_C xyz\, dx$ where C is described by

    $$\mathbf{r}(t) = t\mathbf{i} + (t + 2)\mathbf{j} + (2t - 1)\mathbf{k}, \qquad 0 \le t \le 1$$

    Ans: 1/2

24. Evaluate $\displaystyle\int_C 2xyz\, dx + x^2 z\, dy + x^2 y\, dz$ where C is the curve created by joining the line segments from the origin to (1,0,0), then from (1,0,0) to (1,3,0), then from (1,3,0) to (1,3,2).

    Ans: 6

25. Evaluate the line integral $\int_C x^2y\,dx + (x^3 - y^3)\,dy$ where C is the triangle with vertices (0,0), (2,0), and (1,1).

    Ans: 11/6

26. Identify, sketch, describe, and give rectangular coordinates for the parametric surface given by $\mathbf{r}(u,v) = u\mathbf{i} + 3\cos v\mathbf{j} + 3\sin v\mathbf{k}$ for $0 \le u \le 4$ and $0 \le v \le \pi$.

    Ans: This is the top half of a cylinder of radius 3 (with height 4), rotated about the x-axis. The rectangular coordinates are $z = \sqrt{9 - y^2}$.

27. Evaluate the surface integral $\iint_S (x+y)\,dS$ where S is the surface

$$\mathbf{r}(u,v) = (u\cos v\mathbf{i} + u\sin v\mathbf{j} + (u-1)(2-u)\mathbf{k}$$

    over $0 \le u \le 2$ and $0 \le v \le 2\pi$. You will want to use Sage/Maple to evaluate the surface, and you may want to use them to graph it also to see what the surface looks like.

    Ans: 0

28. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let S be the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{N}\,dS$$

    Use the divergence theorem to set up both integrals, and evaluate the one that you think is easiest.

    Ans: 3

29. Let $\mathbf{F}(x,y,z) = (x-z)\mathbf{i} + (y-z)\mathbf{j} + x^2\mathbf{k}$ and S be the first octant portion of the plane $3x + y + 2z = 12$. Use Stokes' theorem to set up integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ as both a line integral and a double integral, then evaluate whichever you think is easier.

    Ans: 8