Test 2 - MTH 2410

Dr. Graham-Squire, Fall 2012

1:40
2:31
51

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

$180° = \pi$ radians

$1° = \frac{\pi}{180}$ radians

DIRECTIONS

1. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
2. Clearly indicate your answer by putting a box around it.
3. Computers are allowed on one part of this test. Calculators are allowed on all parts of the test except for the last question, for which no technology is allowed. Even on questions where technology is allowed, you should still show all of your work.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge.
6. Number of questions = 11. Total Points = 100.

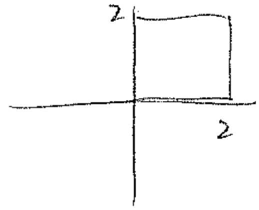
Computer allowed for this question

1. (~~10~~ 2 points) Find the absolute maximum and absolute minimum values for

$$f(x, y) = x^3 - 3xy + y^3$$

in the square region given by $R = \{(x, y) : 0 \le x \le 2 \text{ and } 0 \le y \le 2\}$. You can use Sage/Maple/Grapher/Wolfram Alpha to help you figure out what the answers should be, but all answers need to be supported by correct mathematics.

$f_x = 3x^2 - 3y$ $\qquad$ $f_y = -3x + 3y^2$



$0 = 3(x^2 - y)$ $\qquad$ $0 = 3(y^2 - x)$

$\Rightarrow x^2 = y$ $\qquad$ $x = y^2$

$x = x^4 \Rightarrow x^4 - x = 0 \quad x(x^3 - 1) = 0$

at $x = 0$ and $x = 1$, points $(0,0)$ and $(1,1)$

$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$

at $(0,0)$ $\quad d = 0 \cdot 0 - (-3)^2 < 0 \Rightarrow$ saddlepoint

6

at $(1,1)$ $\quad d = 6 \cdot 6 - (-3)^2 > 0$ and $f_{xx} > 0$

$\Rightarrow (1,1)$ is a minimum

$f(1,1) = 1 - 3 + 1 = -1$

check boundary: $x = 0 \Rightarrow f(y) = y^3$ $\quad$ min of 0 at $y = 0$, max of 8 at $y = 2$

$y = 0 \Rightarrow f(x) = x^3$ get same

$x = 2 \Rightarrow 8 - 6y + y^3 = f(y)$

6

$f'(y) = 3y^2 - 6 = 0$ at $y = \sqrt{2}$

$f(0) = 8$, $f(2) = 6$, $f(\sqrt{2}) = 8 - 6\sqrt{2} + 2\sqrt{2}$

$= 8 - 4\sqrt{2} > 0$, so not less than -1

$y = 2 \Rightarrow x^3 - 6x + 8 = f(x)$

get same as

$$\boxed{\Rightarrow \text{abs. max of } 8 \text{ at } (0, 2, 8), \text{ min of } -1 \text{ at } (1, 1, -1)}$$

Calculators allowed Name: Key

2. (8 points) Let $\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ be an acceleration function. Find the corresponding velocity and position functions given the initial conditions that

$$\mathbf{v}(0) = \mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{r}(0) = \mathbf{i}.$$

$\vec{v}(t)\ \int a(t) = \int(-\cos t\ \vec{i} - \sin t\ \vec{j})\,dt$

$\vec{v}(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{C}_1$

$\vec{j} + \vec{k} = \vec{v}(0) = 0\vec{i} + 1\vec{j} + \vec{C}_1$

$\Rightarrow \vec{C}_1 = \vec{k}$

$\boxed{\text{so } \vec{v}(t) = -\sin t\ \vec{i} + \cos t\ \vec{j} + \vec{k}}$ 4

$\vec{r}(t) = \int \vec{v}\,dt = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k} + \vec{C}_2$

$\vec{i} = \vec{r}(0) = \vec{i} + 0 + 0 + \vec{C}_2$

$\Rightarrow \vec{C}_2 = 0$

$\boxed{\vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}}$ 4

3. (8 points) TRUE OR FALSE. Circle the correct answer. If false, give a counterexample or explain (briefly) why it is false. If true, no explanation is necessary (though if you are wrong, an explanation can get you some partial credit).

(a) **True** or **False:** If f has a relative maximum at (x_0, y_0, z_0), then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

False. You could have a partial that does not exist.

(b) **True** or **False:** The vector-valued function $\langle 4\cos t, 4\sin t, t\rangle$ lies on the cylinder $x^2 + y^2 = 16$.

x y

True, b/c $(4\cos t)^2 + (4\sin t)^2 = 16(\cos^2 t + \sin^2 t)$

$= 16$

x^2 y^2

(c) **True** or **False:** If $\mathbf{r}(t)$ and $\mathbf{u}(t)$ are differentiable vector-valued functions, then $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}'(t)$.

False. Need the ~~[illegible]~~ product rule!

$\frac{d}{dt}[r(t) \cdot u(t)] = r(t) \cdot u'(t) + r'(t) \cdot u(t)$.

(d) **True** or **False:** If the second-order partial derivatives of f exist at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

~~Second partia~~ Mixed partials are always equal.

4. (8 points) Find the curvature K of the curve

$$\vec{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}$$

at the point (-4,0,π).

$\vec{r}'(t) = \langle -4\sin t, 3\cos t, 1\rangle$

$r''(t) = \langle -4\cos t, -3\sin t, 0\rangle$

at $(-4,0,\pi)$, $t=\pi$, so

$\vec{r}'(\pi) = \langle 0, -3, 1\rangle$

$\vec{r}''(\pi) = \langle 4, 0, 0\rangle$

$$K = \frac{\| r'(\pi) \times r''(\pi)\|}{\|r'(t)\|^3}$$

$$\begin{matrix} i & j & k \\ 0 & -3 & 1 \\ 4 & 0 & 0 \end{matrix}$$

$\vec{r}' \times \vec{r}'' = \langle 0, 4, 12\rangle$

$\| \quad \| = \sqrt{160}$

$\|\vec{r}'(t)\| = \sqrt{10}$

$$= \frac{\sqrt{160}}{10\sqrt{10}}$$

$$= \frac{\sqrt{16}}{10}$$

$$= \frac{4}{10} = \boxed{\frac{2}{5}}$$

~~10~~ 10

5. (~~10~~ points) (a) Calculate the limit: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^3-y^3}$.

along $x=0$ get $\lim\limits_{y\to 0} \dfrac{y^3}{-y^3} = \lim\limits_{y\to 0} -1 = -1$

" $y=0$ " $\lim\limits_{x\to 0} \dfrac{x^3}{x^3} = \underline{1}$

5

So limit does not exist because these are not same.

5

(b) Discuss the continuity of the function $g(x,y)$. Justify your answer.

$$g(x,y) = \begin{cases} \dfrac{x^2+2xy^2+y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ \cancel{0}\ \cancel{0}\ 1 & (x,y)=(0,0) \end{cases}$$

~~0~~ ~~0~~ 1

along $x=0$ get $\lim\limits_{y\to 0} \dfrac{y^2}{y^2} = \underline{1}$

~~limit and function don't match that continuous~~

~~along~~ $y=0$ get $\lim\limits_{x\to 0} \dfrac{x^2}{x^2} = \underline{1}$

along $y=x$ get $\lim\limits_{x\to 0} \dfrac{x^2+2x^3+x^2}{x^2+x^2} = \lim\limits_{x\to 0} \dfrac{x^2(2+2x)}{2x^2} = \underline{1}$

do $r = x\cos\theta$ $y = r\sin\theta$

$\Rightarrow \lim\limits_{r\to 0} \dfrac{r^2+2r^3\cos\theta\sin^2\theta}{r^2} = \lim\limits_{r\to 0} 1+2r\cos\theta\sin^2\theta$

$= \underline{1}$

~~So it is continuous at (0,0).~~

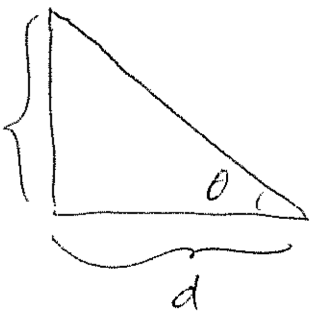
6. (8 points) To determine the height of a tower, the angle of elevation to the top of the tower is measured from a point 100 feet $\pm\frac{1}{2}$ foot from the base. The angle is measured at 33°, with a possible error of 1°. Assuming that the ground is horizontal:

check.

~~(a) Draw a diagram of the situation.~~ (a) For any angle θ and distance d, find an expression for the height of the building $h(d,\theta)$

(b) Use differentials to measure the maximum error in determining the height of the tower.

(a)



$$d = 100 \pm \frac{1}{2} \text{ feet}$$

$$\theta = 33° \pm 1° = 33\left(\frac{\pi}{180}\right) + \frac{\pi}{180}$$

$$\frac{h}{d} = \tan\theta \Rightarrow h = d\tan\theta$$

$$h(d,\theta) = d\tan\theta$$

$$h_d = \tan\theta \qquad h_\theta = d\sec^2\theta$$

$$dh = h_d\, dd + h_\theta\, d\theta \qquad dd = \pm\frac{1}{2}$$

$$d\theta = \pm 1° = 0.02 \text{ radians}$$

$$= h_d(100, 33°)\left(\pm\frac{1}{2}\right) + h_\theta(100, 33°)\left(\pm\frac{\pi}{180}\right)$$

$$= 0.6494\left(\pm\frac{1}{2}\right) + 142.17\left(\pm\pi/180\right)$$

~~max 0.3247 + 142.17 = 142.49~~

$$= \pm 0.3247 \pm 2.4813 \approx \boxed{2.8}$$

With different #s $= 3.278$ feet

10
7. (~~8~~ points) The temperature at the point (x, y) on a metal plate is

$$T = \frac{x}{x^2 + y^2}$$

(a) Find the direction of greatest increase in heat from the point (3,4).

(b) If you are at the point (3,4), how fast is the temperature changing in the direction of the origin (0,0)? Is it increasing or decreasing?

(a) $\nabla T = \left\langle \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2}, \frac{0 - x(2y)}{(x^2+y^2)^2} \right\rangle$

5

$\nabla T(3,4) = \boxed{\left\langle \frac{7}{625}, \frac{-24}{625} \right\rangle}$

(b) (3,4) to (0,0) is vector $\langle -3, -4 \rangle$. ✓

Normalize to get $u = \left\langle \frac{-3}{5}, \frac{-4}{5} \right\rangle$ ✓

$\left\langle \frac{7}{625}, \frac{-24}{625} \right\rangle \cdot \left\langle \frac{-3}{5}, \frac{-4}{5} \right\rangle$ ✓

$= \frac{1}{5(625)}(-21 + 96) = \frac{75}{5(625)} = \boxed{\frac{3}{125}}$ ✓

increasing ✓

8. (8 points) Let $f(x, y) = ye^{xy}$ and $P = (0, 2, 2)$.

(a) Find an equation for the tangent plane to f at P.

(b) Find parametric equations to represent the normal line to f at P.

$f_x = y^2 e^{xy}$ $\quad$ $f_y = e^{xy} + xye^{xy} = e^{xy}(1+xy)$

at $(0,2,2)$: $f_x = 4$, $f_y = 1$

(a) Tangent plane has equation.

$4(x-0) + 1(y-2) - (z-2) = 0$

or $\boxed{4x + y - z = 0}$

(b) gradient has vector $\langle 4, 1, -1 \rangle$, $P = (0,2,2)$

$$\boxed{\begin{aligned} x &= 4t \\ y &= t+2 \\ z &= -t+2 \end{aligned}}$$

9. (8 points) Calculate $\partial w/\partial r$ and $\partial w/\partial \theta$ if

$$w = x^2 - 2xy + y^2, \qquad x = r + \theta, \qquad \text{and } y = r - \theta.$$

$$\frac{\partial w}{\partial x} = 2x - 2y \qquad \frac{\partial w}{\partial y} = 2y - 2x \qquad \frac{\partial x}{\partial r} = 1 \qquad \frac{\partial x}{\partial \theta} = 1$$

$$\frac{\partial y}{\partial r} = 1 \qquad \frac{\partial y}{\partial \theta} = -1$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = (2x-2y)(1) + (2y-2x)\,1 = \boxed{0}$$

$$\frac{\partial w}{\partial \theta} = (2x-2y)\,1 + (2y-2x)(-1) = 4x - 4y$$

$$= 4(r+\theta) - 4(r-\theta)$$

$$= \boxed{8\theta}$$

Extra Credit(2 points) If you wanted to visualize the function

$$f(x, y, z) = 3x^2 - 4yz + z^2x^3,$$

how would you do it? Explain in words and/or math notation.

I would set $f(x,y,z) = c$ and look at level surfaces to see how the function changes through them.

If you need more room for the last question, you can write here.

No Calculator Name:______________________

10. (10 points) Let a particle move along the path

$$\mathbf{r}(t) = \langle \cos(\pi t) + \pi t \sin(\pi t), \sin(\pi t) - \pi t \cos(\pi t) \rangle.$$

(a) Find expressions for $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Hint: The derivatives are a little tricky, but some things should cancel out and when you calculate the norm of $\mathbf{r}'(t)$ it should simplify quite a bit.

(b) A graph of $\mathbf{r}(t)$ is given below. Find the position for the particle at $t = 2$, and sketch the vectors $\mathbf{T}(2)$ and $\mathbf{N}(2)$ with their initial points at the position given by $t = 2$.

(a) $r'(t) = \langle \cancel{-\pi \sin(\pi t)} + \cancel{\pi \sin(\pi t)} + \pi^2 t \cos(\pi t), \cancel{\pi \cos(\pi t)} - (\cancel{\pi \cos(\pi t)} - \pi^2 t \sin(\pi t)) \rangle$

$= \langle \pi^2 t \cos(\pi t), \pi^2 t \sin(\pi t) \rangle = \pi^2 t \langle \cos(\pi t), \sin(\pi t) \rangle$

$\|r'(t)\| = |\pi^2 t| \cdot 1$

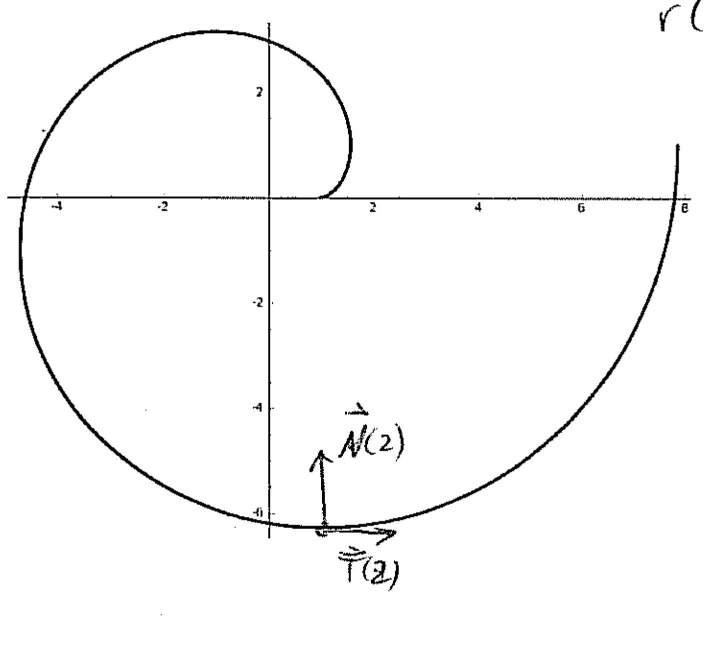
$\Rightarrow \vec{T}(t) = \dfrac{r'(t)}{\|r'(t)\|} = \boxed{\langle \cos(\pi t), \sin(\pi t) \rangle = \vec{T}(t)}$ ✓✓✓✓✓

$T'(t) = \langle -\pi \sin(\pi t), \pi \cos(\pi t) \rangle = \pi \langle -\sin(\pi t), \cos(\pi t) \rangle$

$\|T'(t)\| = \pi$ $\Rightarrow \boxed{\vec{N}(t) = \langle -\sin(\pi t), \cos(\pi t) \rangle}$ ✓✓

(b) $T(2) = \langle 1, 0 \rangle$, $N(2) = \langle 0, 1 \rangle$

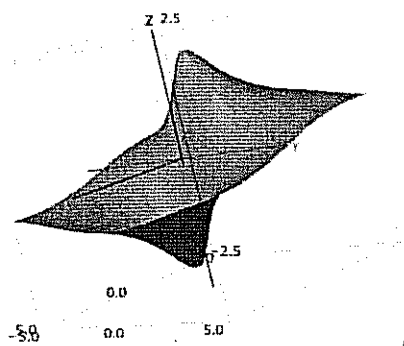
$r(2) = \langle 1+0, 0-2\pi \rangle = \langle 1, -2\pi \rangle$



✓✓✓

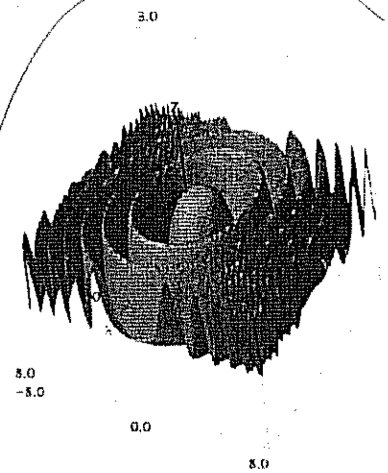
11. (8 points) Match the function on this page with the graph of its level curves on the next page.

(h)



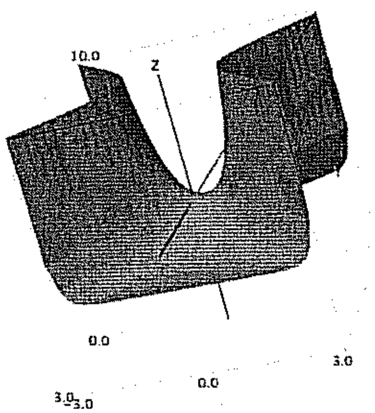
(a) $f(x,y) = \dfrac{-4x}{x^2+y^2+1}$

(g)



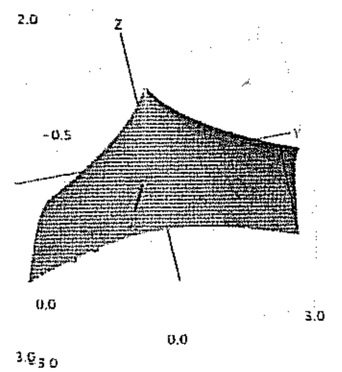
(b) $f(x,y) = \cos\left(\dfrac{x^2+2y^2}{4}\right)$

(e)

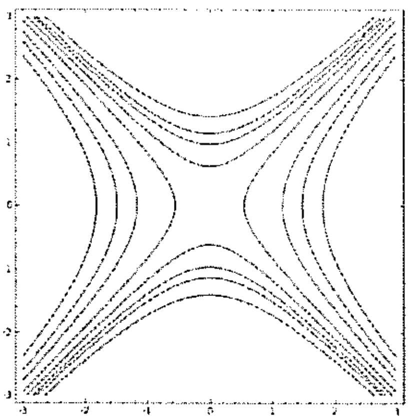


(c) $f(x,y) = e^{1-x^2+y^2}$

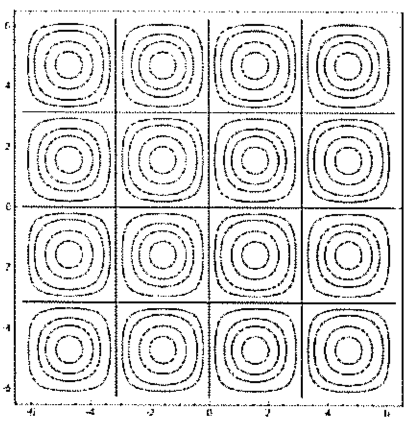
(i)



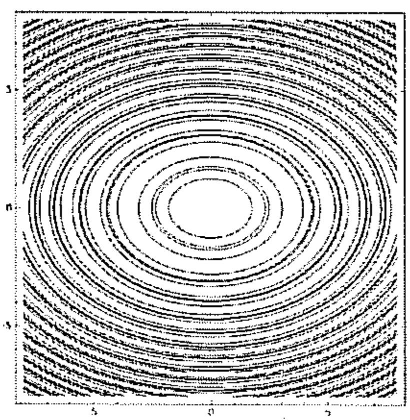
(d) $f(x,y) = 1 - (x^2+y^2)^{1/3}$



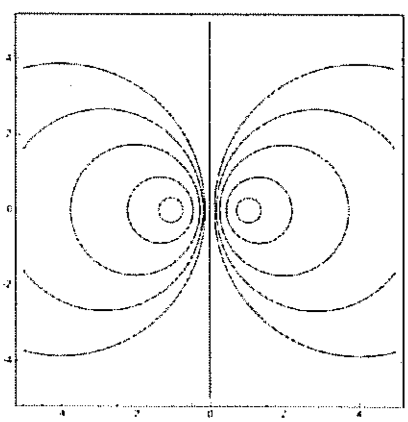
(e)



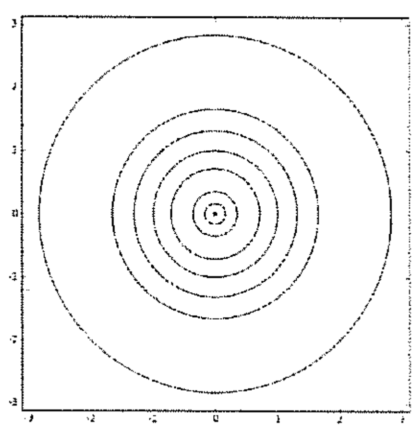
(f)



(g)



(h)



(i)