Calculus III, Final Exam Review

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•The final will cover all sections from 11.1 to 15.8, though some sections may not have a question specifically targeted for them. For example, a question on Stokes' theorem may involve you taking a curl, finding a gradient, visualizing a surface, parametrizing a curve, and evaluating a surface integral. So that one question may cover multiple sections in just one question.

•To study, you can look over your notes, rework HW problems on WebAssign, quizzes, and problems from the notes, as well as work out the practice problems given for each section. The Review Questions at the end of each chapter may also be useful, as the odd numbered questions have the answers in the back of the book.

•Calculators are allowed on most of the final exam, but for certain questions you may not be allowed to use a calculator. It is highly recommended that you bring a calculator because you cannot use cell phones or computers during most of the test. There will also be at least one question where you can use a computer.

•Some practice problems to work on are below. I have not covered all sections in depth with this review material, it is just meant to capture those questions that I think are most relevant. In other words, you should not think that just doing these review questions will be enough to prepare you for the final exam.

1. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.
   (a) $\mathbf{u} = \langle -4, 3, -6\rangle, \quad \mathbf{v} = \langle 16, -12, 24\rangle$.
   (b) $\mathbf{u} = \langle -4, 3, -6\rangle, \quad \mathbf{v} = \langle 5, 9, 1\rangle$.
2. Find a set of parametric equations for the line given by the intersection of the planes $3x - 3y - 7z = -4$ and $x - y + 2z = 3$. (Hint: find the normal vectors for each of the given planes. The cross of the normals will give you the direction vector for the line, and then you just need to find a point that lies on both of the planes.)
3. Describe and sketch the surfaces:
   (a) $y = \cos x$
   (b) $16x^2 + 16y^2 - 9z^2 = 0$
   (c) $v\mathbf{i} + 2\cos u\mathbf{j} + 2\sin u\mathbf{k}$
4. Convert the rectangular equation $x^2+y^2+z^2 = 16$ to an equation in (a) cylindrical coordinates and (b) spherical coordinates.
5. Sketch and describe the space curve given by the intersection of the plane $x - y = 0$ and the surface $x^2 + z^2 = 4$. Use the parameter $x = t$ to find a vector-valued function for the space curve.
6. Evaluate the limit: $\displaystyle\lim_{t\to 0}\left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^{t}\mathbf{k}\right)$.

7. For the vector-valued functions $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$ and $\mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{t}\mathbf{k}$, find
   (a) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$
   (b) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$
   (c) $D_t[\mathbf{u}(t) - 2\mathbf{r}(t)]$

8. A projectile is fired from ground level at tan angle of 20° with the horizontal. The projectile has a range of 95 feet. Find the minimum initial velocity.

9. Find the curvature K of $\mathbf{r}(t) = \langle 2t, 5\cos t, 5\sin t\rangle$.

10. Find the limit and discuss the continuity of the function: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$.

11. For $f(x,y) = \cos(x-2y)$, find all second partial derivatives and confirm that the mixed partials are equal.

12. A right circular cone is measured and the radius and height are found to be $r = 2in$ and $h = 5in$. The possible error in each measurement is $\frac{1}{8}$ inch. Use differentials to approximate the maximum possible error in the calculation of the volume ($V = \frac{1}{3}\pi r^2 h$).

13. A team of oceanographers is mapping topography to assist in the recovery of a crashed helicopter. They develop the model

$$H = 250 + 30x^2 + 50\sin\frac{\pi y}{2}, \qquad 0 \le x \le 2, 0 \le y \le 2$$

   where H is height in meters (above sea level), and x and y are distances in *kilometers*.
   (a) What is the height of the helicopter if it is located at the point $x = 1$ and $y = 0.5$?
   (b) What is the direction of greatest steepness from the location of the helicopter?
   (c) How steep is the slope if you go in the direction of greatest steepness?
   (d) How steep is the slope if you travel from the helicopter in the direction of the point (2,2)?

14. Find an equation for the tangent line and parametric equations of the normal line to the surface $f(x,y) = \sqrt{25-y^2}$ at the point (2,3,4).

15. Examine the surface $z = 50(x+y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$ for relative extrema and saddle points. Identify all such points, then use a computer to graph the surface and check your work.

16. The production function for a candy manufacturer is $f(x,y) = 4x + xy + 2y$, where x is the number of units of labor and y is the number of unit of capital. Assume that units of labor cost \$20 and units of capital cost \$4, and the total amount of money available for both labor and capital is \$2000. Write a constraint equation and then find the maximum production level for this manufacturer.

17. Evaluate the iterated integral. Change order of integration or coordinates as needed.
   (a) $\displaystyle\int_0^2 \int_{x^2}^{2x} (x^2 + 2y)\, dy\, dx$
   (b) $\displaystyle\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\, dy\, dx$

18. Find the volume of the solid:

    (a) bounded by the graphs of $z = x + y$, $z = 0$, $y = 0$, $x = 3$, and $y = x$.

    (b) bounded by the graphs of $z = 0$ and $z = 4$, outside the cylinder $x^2 + y^2 = 1$ and inside the hyperboloid $x^2 + y^2 - z^2 = 1$. You can use a computer to help visualize the region, but you should be able to do the integral by hand.

19. Find the area of the surface $f(x) = 4 - x^2$ over the region given by the triangle bounded by the graphs of $y = x$, $y = -x$, and $y = 2$. You can use a computer to integrate the integral.

20. Evaluate the integral $\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}\, dz\, dy\, dx$. Hint: once you have set up the integral correctly, it may be helpful to rewrite the integrand as $1 - \frac{1}{\text{something}}$.

21. Find the center of mass of the solid bounded below by $x^2 + y^2 + z^2 = 25$ and above by $z = 4$. Assume that the density is constant.

22. Use the change of variables $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$ to evaluate the double integral $\iint\limits_R \ln(x + y)\, dA$ where R is the square region with corners at (1,2), (2,1), (3,2) and (2,3).

23. Evaluate the line integral $\int_C xyz\, dx$ where C is described by

$$\mathbf{r}(t) = t\mathbf{i} + (t + 2)\mathbf{j} + (2t - 1)\mathbf{k}, \qquad 0 \le t \le 1$$

24. Evaluate $\int_C 2xyz\, dx + x^2 z\, dy + x^2 y\, dz$ where C is the curve created by joining the line segments from the origin to (1,0,0), then from (1,0,0) to (1,3,0), then from (1,3,0) to (1,3,2).

25. Evaluate the line integral $\int_C x^2 y\, dx + (x^3 - y^3)\, dy$ where C is the triangle with vertices (0,0), (2,0), and (1,1).

26. Identify, sketch, describe, and give rectangular coordinates for the parametric surface given by $\mathbf{r}(u, v) = u\mathbf{i} + 3\cos v\mathbf{j} + 3\sin v\mathbf{k}$ for $0 \le u \le 4$ and $0 \le v \le \pi$.

27. Evaluate the surface integral $\iint\limits_S (x + y)\, dS$ where S is the surface

$$\mathbf{r}(u, v) = (u\cos v\mathbf{i} + u\sin v\mathbf{j} + (u - 1)(2 - u)\mathbf{k}$$

    over $0 \le u \le 2$ and $0 \le v \le 2\pi$. You will want to use Sage/Maple to evaluate the surface, and you may want to use them to graph it also to see what the surface looks like.

28. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let S be the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. Evaluate

$$\iint\limits_S \mathbf{F} \cdot \mathbf{N}\, dS$$

    Use the divergence theorem to set up both integrals, and evaluate the one that you think is easiest.

29. Let $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$ and S be the first octant portion of the plane $3x + y + 2z = 12$. Use Stokes' theorem to set up integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ as both a line integral and a double integral, then evaluate whichever you think is easier.