Test 2A - MTH 1420

Dr. Graham-Squire, Spring 2013

3:25

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

1. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
2. Clearly indicate your answer by putting a box around it.
3. Cell phones are not allowed on this test. Computers and/or calculators are allowed on the first 2 questions. Calculators (but not computers) are allowed on questions 3-6, however you should still show all of your work. No calculators are allowed on the last 3 questions of the test.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. If you have a problem involving trigonometric substitution or integration, these may help:
   - If we have a factor of the form $\sqrt{a^2 - x^2}$, we do the substitution $x = a\sin\theta$.
   - If we have a factor of the form $\sqrt{a^2 + x^2}$, we do the substitution $x = a\tan\theta$.
   - If we have a factor of the form $\sqrt{x^2 - a^2}$, we do the substitution $x = a\sec\theta$.

   - $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
   - $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
6. Make sure you sign the pledge.
7. Number of questions = 9. Total Points = 90.

Calculator and Computers are okay

Simplify!

1. (10 points) Find the arc length of the function $y = e^x \sin x$ on the interval $2 \leq x \leq 6$. If you cannot evaluate the integral by hand, use a calculator or Sage/Maple to do numerical integration. Round to nearest 0.01

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \qquad \frac{dy}{dx} = e^x \cos x + e^x \sin x$$

$$= \int_2^6 \sqrt{1 + (e^x \cos x + e^x \sin x)^2}\, dx$$

$$= \int_2^6 \sqrt{1 + (e^x)^2 (\cos^2 x + 2\cos x \sin x + \sin^2 x)}\, dx$$

$$= \int_2^6 \sqrt{1 + e^{2x}(1 + 2\cos x \sin x)}\, dx$$

Cannot go any further. Calculator gives

$$= 241.00$$

8

2

(S_6)

2. (10 points) Use Simpson's Rule with <u>six</u> subintervals to approximate the arc length of the function $y = e^x \sin x$ on the interval $2 \le x \le$ ~~4~~ 5. That is, use Simpson's rule to approximate the integral you set up in question 1.

Round to nearest 0.01

Simpson's Rule is given by

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

where n is even, $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

$$\Delta x = \frac{5-2}{6} = \frac{3}{6} = \frac{1}{2} \qquad f(x) = \sqrt{1+e^{2x}(1+2\cos x \sin x)}$$

$$S_6 = \frac{\left(\frac{1}{2}\right)}{3}\left(f(2)+4f(2.5)+2f(3)+4f(3.5)+2f(4)+4f(4.5)+f(5)\right)$$

$$= \frac{1}{6}\left(3.78+4(2.66)+2(17.08)+4(42.64)+2(77.01)+4(106.97)+100.22\right)$$

$$= 150.21$$

Calculators are okay, no computers allowed

Test A

3. (10 points) Three integrals are given below. For only <u>one</u> of them is it necessary to do trigonometric substitution.

(i) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sqrt{x^2-9}}\,dx$ (ii) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{x^2+9}{x^2}\,dx$ (iii) $\displaystyle\int_{\cancel{\pi/6}\ 3/2}^{\cancel{\pi/2}\ 3} \sqrt{9-x^2}\,dx$

(a) Which of the two do <u>not</u> require trigonometric substitution? For each, explain how you would integrate it (i.e., what integration method you would use to solve it), but **do not actually do the integration.**

(b) Use trigonometric substitution to integrate the one that needs trigonometric substitution.

4

(a) (i) does not need trig subs, just regular substitution with $u = x^2-9$.

(ii) does not need trig subs, just rewrite as $\displaystyle\int_{\pi/6}^{\pi/2}\left(1+\frac{9}{x^2}\right)dx$ $\to 9x^{-2}$

3

(b) For (iii), do $x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

x	θ
$\cancel{\pi/2}$ $3/2$	$\pi/6$
3	$\pi/2$

$\frac{1}{2} = \sin\theta$

$1 = \sin\theta$

$$\Rightarrow \int_{\pi/6}^{\pi/2} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 9\cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int_{\pi/6}^{\pi/2} (1+\cos(2\theta))\,d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\Big|_{\pi/6}^{\pi/2}\right)$$

$$= \frac{9}{2}\left(\frac{\pi}{2} + \frac{\cancel{\sin\pi}}{2} - \left(\frac{\pi}{6}+\frac{\sin\pi/3}{2}\right)\right)$$

$$= \frac{9}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$\frac{3\,\cancel{9}}{2}\left(\frac{4\pi - 3\sqrt{3}}{\cancel{12}\,4}\right)$$

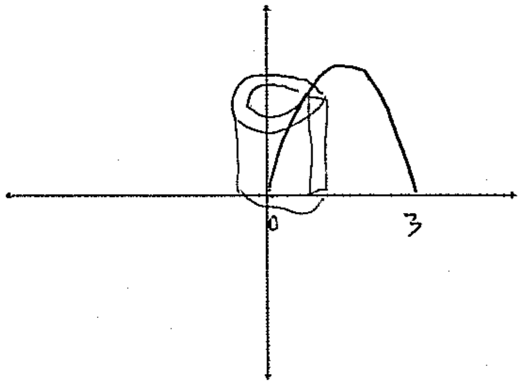
$$\boxed{\frac{12\pi - 9\sqrt{3}}{8}}$$

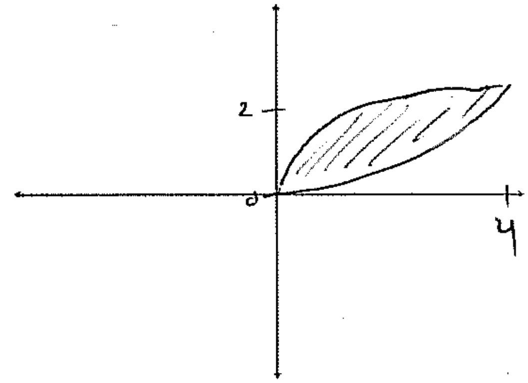
4. (10 points) Below, two solids of revolution are described. You should choose one of them and set up the integral to represent how to find the volume of the solid. **You do not need to set up both integrals, only one is necessary. If you set up both, you must circle which one you want me to grade.**

(i) The region R in the first quadrant enclosed by the curve $y = -x^4 + 3x^3$ and the x-axis, rotated about the y-axis.

(ii) The region S in the first quadrant enclosed by the curves $y^2 = x$ and $y = \frac{x^2}{8}$, rotated about the x-axis.

(i)





$y = -x^3(x-3)$

use shells.

$$\boxed{\int_0^3 2\pi\, x(-x^4 + 3x^3)\, dx}$$

$8y = y^4$

$0 = y(y^3 - 8) \Rightarrow y = 0, 2$

$x = 4$

or

Shells:

$$2\pi \int_0^2 (y)(\sqrt{8y} - y^2)\, dy$$

or

Washers:

$$\pi \int_0^4 \left((\sqrt{x})^2 - \left(\frac{x^2}{8}\right)^2 \right) dx$$

-3 for wrong variable

-1.5 for wrong top/bottom order.

3:44 → 19 min.

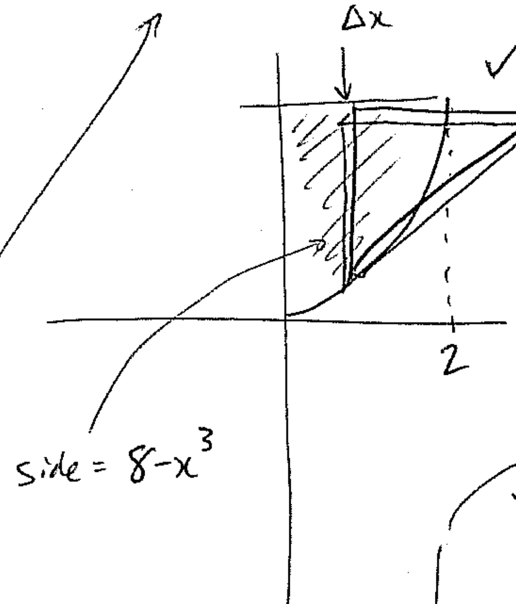
3:46
3:57
11

5. (10 points) Let S be the region in the first quadrant bounded by the lines $x = 0$, $y = 8$, and $y = x^3$. Find the volume of the solid with base S such that cross-sections perpendicular to the <u>x-axis</u> are isosceles right triangles with one leg (not the hypotenuse) lying on S.

$8 = x^3$

$x = 2$

$\Delta = \frac{1}{2} b \cdot h$



axes

$$V_i = \Delta x\,(\text{side})^2 \cdot \frac{1}{2}$$
$$= \Delta x\,(8-x^3)^2 \cdot \frac{1}{2}$$

$$\Rightarrow \text{Volume} = \int_0^2 \frac{1}{2}(8-x^3)^2\,dx$$

$$= \frac{1}{2}\int_0^2 (64 - 16x^3 + x^6)\,dx$$
$$= \frac{1}{2}\left(64x - 4x^4 + \frac{x^7}{7}\Big|_0^2\right)$$
$$= \frac{1}{2}\left(128 - 64 + \frac{128}{7}\right)$$
$$= 32 + \frac{64}{7}$$
$$= \frac{224-64}{7} = \boxed{\frac{160}{7}}$$

Set up but do <u>not</u> integrate.

6. (10 points) Are the following integrals improper or not? For each improper integral, express it using the appropriate limit notation. If the integral is *not* improper, just say 'not improper'. **Do not actually integrate any of these!**

(i) $\int_0^{3\pi/4} \tan x\, dx$ Improper because $\tan\left(\frac{\pi}{2}\right)$ is undefined!

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$$\int_0^{3\pi/4} \tan x\, dx = \lim_{b \to \pi/2^-} \int_0^b \tan x\, dx + \lim_{a \to \pi/2^+} \int_a^{3\pi/4} \tan x\, dx$$

(ii) $\int_2^7 \sin(\ln x)\, dx$

Not improper

2

(iii) $\int_0^3 \frac{x-1}{x^2-6x+9}\, dx = \int_0^3 \frac{x-1}{(x-3)^2}\, dx$ undefined at $x=3$.

3

$$= \lim_{b \to 3^-} \int_0^b \frac{x-1}{(x-3)^2}\, dx$$

(iv) $\int_0^4 \frac{x-4}{e^x}\, dx$

Not improper.

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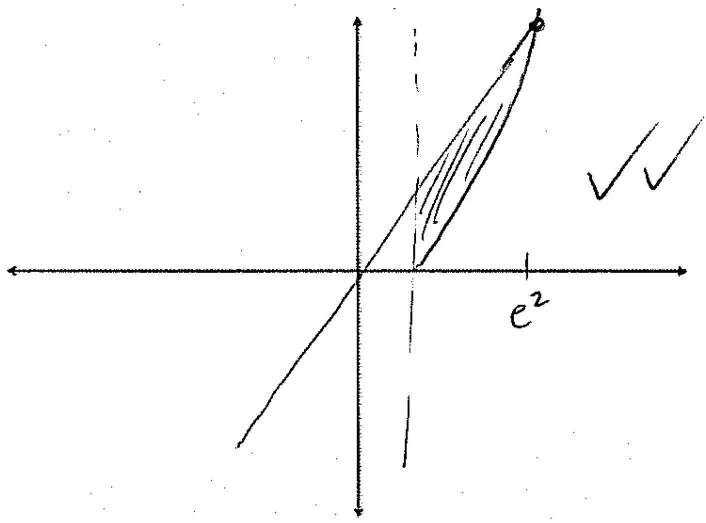
No Calculator

Test A

7. (10 points) Find the area bounded by $y = x \ln x$, $y = 2x$, and the line $x = 1$. Leave your answer in exact form.



$2x = x \ln x$

$2 = \ln x$

$e^2 = x$

$$\int_1^{e^2} (2x - x \ln x)\,dx$$ 4

$$= \int_1^{e^2} 2x\,dx - \int_1^{e^2} x \ln x\,dx$$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{1}{2}x^2$

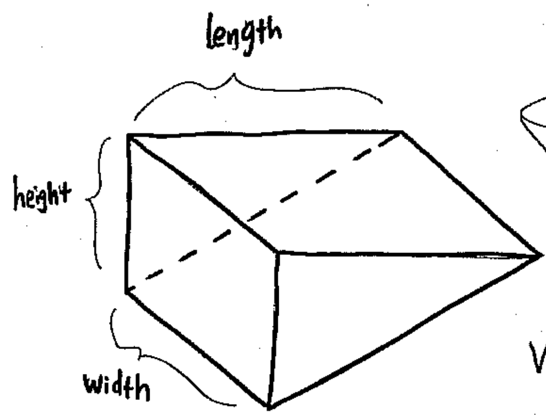
$$= x^2\Big|_1^{e^2} - \left(\frac{1}{2}x^2 \ln x\Big|_1^{e^2} - \int_1^{e^2} \frac{1}{2}x\,dx\right)$$

$$= e^4 - 1 - \frac{1}{2}\left(e^4 \ln e^2 - 0\right) + \frac{1}{4}x^2\Big|_1^{e^2}$$

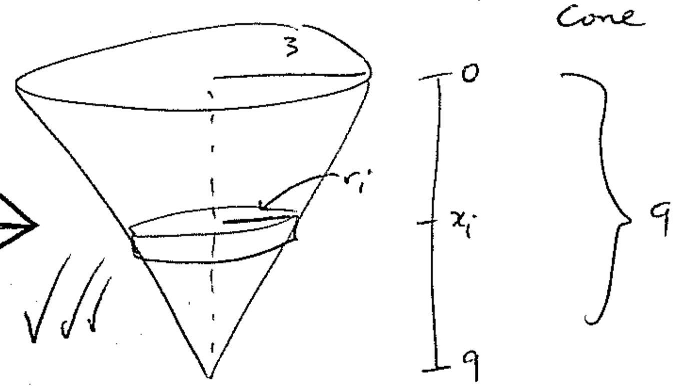
$$= \cancel{e^4} - 1 - \cancel{e^4} + \frac{1}{4}e^4 - \frac{1}{4}$$

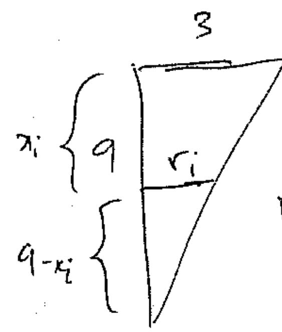
$$\boxed{\frac{1}{4}e^4 - \frac{5}{4}}$$

8. (10 points) A trough filled with water is in the shape of a triangular prism, with the pointy end down as shown in the picture. The triangle is a right triangle with length 12 ft and height 6 ft, and the trough is 5 ft wide. Set up, but **do not integrate**, an integral to find the amount of work needed to pump all of the water out of the top of the tank.



Replace w/ ~~Stack~~ Cone





$$\frac{3}{9} = \frac{r_i}{9-x_i}$$

$$\frac{1}{3}(9-x_i) = r_i$$

$$\Rightarrow V_i = \pi\left(3-\frac{x_i}{3}\right)^2 \cdot \Delta x$$

$$\Rightarrow \text{Force}_i = \pi\left(3-\frac{x_i}{3}\right)^2 \Delta x \cdot 62.5$$

$$W_i = \pi\left(3-\frac{x_i}{3}\right)^2 \Delta x \cdot 62.5 \cdot (x_i)$$

$$\Rightarrow \boxed{\text{Work} = \int_0^9 62.5\pi \left(3-\frac{x}{3}\right)^2 \cdot x \, dx}$$

9. (10 points) Is the integral

$$\int_3^\infty \frac{1}{(x-1)^3}\,dx$$

convergent or divergent? If convergent, what is the value of the integral? Make sure to use correct mathematical notation.

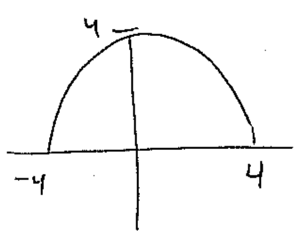
$$\lim_{b\to\infty}\int_3^b \frac{1}{(x-1)^3}\,dx = \lim_{b\to\infty} \left.\frac{(x-1)^{-2}}{-2}\right|_3^b$$

$$= \lim_{b\to\infty} \frac{(b-1)^{-2}}{-2} - \frac{(2)^{-2}}{-2}$$

$$= 0 + \frac{1}{8}$$

$$= \boxed{\frac{1}{8}}$$

Extra Credit(2 point) Either graph and use geometry, or use the arc length formula, to find the arc length of the parametric curve $x = 4\cos\theta$, $y = 4\sin\theta$ for $0 \le x \le \pi$.



= half a circle

$= \pi \cdot \text{radius} = \boxed{4\pi}$