Test 3A - MTH 1410

and 3C

Name: Key

PID Number: ____________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

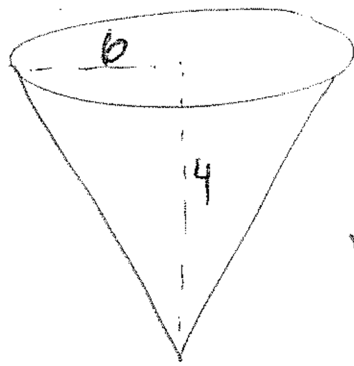
1. Show all of your work. A correct answer with insufficient work will be counted wrong.
2. Clearly indicate your answer by putting a box around it.
3. Calculators are allowed on this exam.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge and write your PID on both pages.
6. Number of questions = 10. Total Points = 100.

PID Number: Key

Test 3A/3C 1:31

1. (12 points) Water is pouring into an inverted cone-shaped tank. The cone has a radius of 6 meters and a height of 4 meters. Note that the cone is <u>inverted</u>, so the base of the cone is actually the top of the tank. When the depth of the water is 2 meters, the depth is increasing at a rate of 0.2 meters/second. How fast is the volume of the water changing at that instant? (Note: The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, and you will need to use the fact that the radius equals 3/2 times the height).

$\frac{dh}{dt} = 0.2\ m/s$



Want $\left.\frac{dV}{dt}\right|_{h=2}$ $\qquad r = \frac{3}{2}h$

$$\Rightarrow V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h$$

$$\frac{d}{dt}\left(V = \frac{3}{4}\pi h^3\right)$$

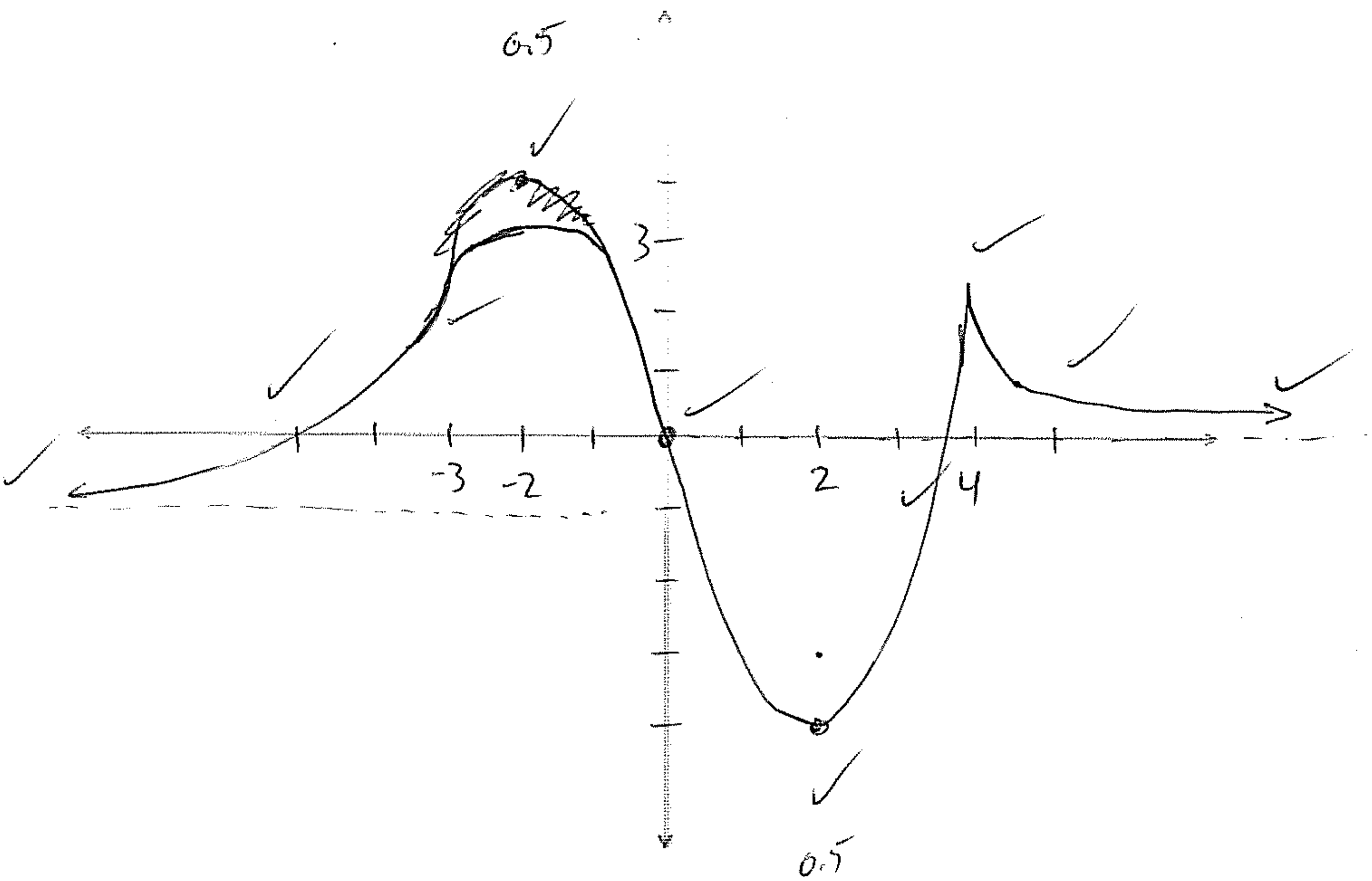
$$\Rightarrow \frac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$= \frac{9}{4}\pi \cdot (2)^2 \cdot 0.2$$

$$\boxed{\frac{dV}{dt} = 1.8\pi}$$

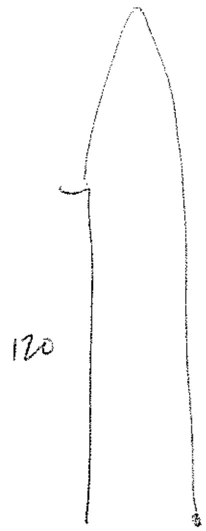
2. (12 points) Sketch the graph of a <u>continuous</u> function $f(x)$ given the following information:

- $f(0) = 0$, the absolute minimum value of f is -4, and the absolute maximum value of f is 3.
- $f'(-3)$ and $f'(4)$ do not exist.
- $f'(x) > 0$ on the intervals $(-\infty, -3), (-3, -2)$ and $(2, 4)$.
- $f'(x) < 0$ on the intervals $(-2, 2)$ and $(4, \infty)$.
- $f''(x) > 0$ on the intervals $(-\infty, -3)$ and $(0, 4)$ and $(4, \infty)$.
- $f''(x) < 0$ on the interval $(-3, 0)$
- $\lim_{x \to (-\infty)} f(x) = -1$ and $\lim_{x \to \infty} f(x) = 0$



3. (10 points) On the surface of the moon, an observation tower is built with a catapult on top. The tower is 120 meters high, and the catapult shoots a ball off the side of the tower. Assuming acceleration due to gravity on the moon is -4 m/sec^2, and the ball has initial upward velocity of 22 m/sec, how long will it take for the ball to hit the surface of the moon?

a constant



$\Rightarrow s'(0) = 22$

$a(t) = s''(t) = -4$

$s'(t) = -4t + C$

$22 = s'(0) = -4 \cdot 0 + C = C$

So $s'(t) = -4t + 22$

$\Rightarrow s(t) = -2t^2 + 22t + D$

$s(0) = 120$

$\Rightarrow 120 = 0 + 0 + D$

$\Rightarrow s(t) = -2t^2 + 22t + 120$

Want $0 = -2t^2 + 22t + 120$

$0 = t^2 - 11t - 60$

$0 = (t-15)(t+4)$ $\to t = -4$ $\to t = 15$

Not in domain.

$\Rightarrow$ $\boxed{\text{hits after } 15 \text{ sec}}$

4. (6 points) Find the most general antiderivative of

$$f'(x) = \frac{x^6 - 2x^3 + 1}{x^4}$$

$$f'(x) = \frac{x^6}{x^4} - \frac{2x^3}{x^4} + \frac{1}{x^4}$$

$$f'(x) = x^2 - 2 \cdot \frac{1}{x} + x^{-4}$$

$$\Rightarrow f(x) = \boxed{\frac{x^3}{3} - 2\ln|x| - \frac{1}{3}x^{-3} + C}$$

5. (12 points) Consider the function $f(x) = \dfrac{x}{3x^2+12}$.

(a) Find the x-coordinate(s) where f has a local maximum or a local minimum. Clearly identify if your answer is a local maximum or a local minimum, and show your work.

$$f'(x) = \frac{1(3x^2+12) - x(6x)}{3x^2+12}$$

$$\Rightarrow f'(x) = \frac{-3x^2+12}{3x^2+12} = \frac{-3(x^2-4)}{3x^2+12} = \frac{-3(x-2)(x+2)}{3x^2+12}$$

$0 = -3(x-2)(x+2)$ when $x = -2$ and $x = 2$

$f'(-3)<0$ $\quad$ $f'(0)>0$ $\quad$ $f'(3)<0$

$- - - - -$ $\quad$ $+ + + + + +$ $\quad$ $- - - -$

-2 $\quad$ 2

$\boxed{\begin{array}{l} x=-2 \text{ is a local min} \\ x=2 \text{ is a local max} \end{array}}$

(b) Find the absolute maximum of f on the interval $[-4, 0]$.

$f(-4) = \dfrac{-4}{60}$

$f(-2) = \dfrac{-2}{24}$

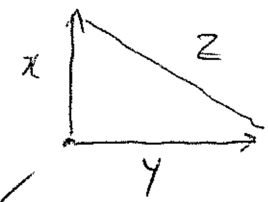
$f(0) = \boxed{0}$

3

6. (8 points) Two cars leave from the same point at the same time. One travels north at 21 miles/hour and the other travels east at 28 miles/hour. Use calculus to find the rate at which the two cars are moving away from each other after two hours.

$\frac{dx}{dt} = 21$

$\frac{dy}{dt} = 28$



$\frac{d}{dt}\left(x^2 + y^2 = z^2\right)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$\cancel{2}(42)\cdot 21 + \cancel{2}(56)\cdot 28 = \cancel{2}(70)\frac{dz}{dt}$

$\frac{42\cdot \overset{3}{\cancel{21}} + 56\cdot \cancel{28}\,4}{\cancel{70}\ 10} = \frac{dz}{dt}$

After 2 hours

$x = 42 \quad (3\cdot 14)$

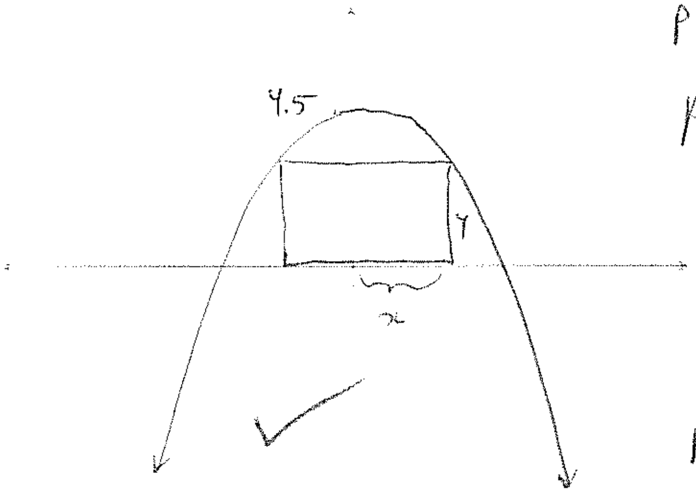
$y = 56 \quad (4\cdot 14)$

$z = 70 \quad (5\cdot 14)$

$\frac{126 + 224}{10} = \frac{dz}{dt}$

$\boxed{35\,\text{m/h} = \frac{dz}{dt}}$

7. (10 points) A rectangle with its bottom on the x-axis has upper corners on the parabola $y = 4.5 - \frac{1}{2}x^2$. Find the maximum perimeter possible for such a rectangle. Be sure to check that your answer is a maximum.



$P = 2x + y + 2x + y$

$P = 4x + 2y$

$P(x) = 4x + 2(4.5 - \frac{1}{2}x^2)$

$P(x) = -x^2 + 4x + 9$

$P'(x) = -2x + 4$

$0 = -2x + 4$

$x = 2$

$P''(x) = -2 \Rightarrow$ concave down, so $x = 2$ is a max.

$\Rightarrow P(x) = -4 + 8 + 9$

$= \boxed{13}$

8. (8 points) For the following statements, answer True or False. If True, give a brief explanation of why. If False, either explain why or give a counterexample. A picture may help, but needs some words to explain it.

(a) "Suppose I wanted to use Newton's Method on $f(x) = 2x^3 - 24x + 31$. Since $f(2) = -1$, $x_1 = 2$ would be a good first guess."

False ✓✓

$f'(x) = 6x^2 - 24$

$\Rightarrow f'(2) = 0$ so Newton's Method would not work.

✓✓

(b) "For a continuous function defined on a closed interval, the absolute maximum and absolute minimum values must occur at either the endpoints of the interval or a critical number." ✓✓

Closed Interval = $[a, b]$

True. To find abs. max or min you find critical numbers c, then do

$f(a) =$

$f(c) =$

$f(b) =$ and compare.

✓

9. (12 points) Calculate the following limits. If the limit does not exist, explain why.

(a) $\lim_{x\to 0} \dfrac{\ln(1+x)}{\tan(2\pi x)}$ $\longrightarrow \frac{0}{0}$ ✓

$$\overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{\sec^2(2\pi x)\cdot 2\pi}$$ ✓✓✓

$$= \lim_{x\to 0} \frac{\frac{1}{1+x}}{\frac{2\pi}{\cos^2(2\pi x)}}$$ ✓

$$= \lim_{x\to 0} \frac{1}{1+x} \cdot \frac{\cos^2(2\pi x)}{2\pi}$$ ✓

$$= \frac{1}{1} \cdot \frac{1}{2\pi} = \boxed{\frac{1}{2\pi}}$$

(b) $\lim_{x\to 0^+} \dfrac{e^x}{x^3 - 2x}$

$$= \frac{e^0}{0-0} = \frac{1}{0} \Rightarrow \infty \quad \text{or} \quad -\infty$$ (4)

as $x \to 0^+$, $e^x > 0$ and $x^3 - 2x = x(x^2-2)$

$+ \cdot -$ is negative.

get $\frac{+}{-} \Rightarrow \boxed{-\infty}$ ✓✓

$2x^2 - x - 3 = 0$

10. (10 points) Given the equation $2x^2 - x = 3$, use Newton's Method to find a solution for x. Use a first guess of $x_1 = 1$ and find the solution accurate to the first two decimal places.

$f(x) = 2x^2 - x - 3$

$f'(x) = 4x - 1$

$x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{3} = \frac{5}{3}$$

$$x_2 = 1.66\overline{6}$$

$$x_3 = 1.6\overline{6} - \frac{f(1.\overline{6})}{f'(1.\overline{6})} = \underbrace{1.50}98039$$

$$x_4 = 1.5098 - \frac{f(1.5098)}{f'(1.5098)} = \underbrace{1.50}004$$

$$\boxed{1.50}$$

Extra Credit(2 points) Find an antiderivative of $f'(x) = e^e$.

$f(x) = e^e \cdot x$

Test 3B - MTH 1410 and 3D

Name: Key

PID Number:

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(*signature*)

Fix #9

Third dot ⇒ $(-4,-2)$

DIRECTIONS

1. Show all of your work. A correct answer with insufficient work will be counted wrong.
2. Clearly indicate your answer by putting a box around it.
3. Calculators <u>are</u> allowed on this exam.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge and write your PID on both pages.
6. Number of questions = 9. Total Points = 100.

PID Number: Key

Test 3B/3D

1. (10 points) On the surface of the moon, an observation tower is built with a catapult on top. The tower is 80 meters high, and the catapult shoots a ball off the side of the tower. Assuming acceleration due to gravity on the moon is -4 m/sec^2, and the ball has initial upward velocity of 12 m/sec, how long will it take for the ball to hit the surface of the moon?

80 m. high $\Rightarrow p(0) = 80$ ✓

$v(0) = 12$ ✓

$a(t) = -4$ ✓

$v(t) = -4t + C$

$12 = v(0) = -4 \cdot 0 + C$ ✓

$\Rightarrow C = 12$

$\Rightarrow v(t) = -4t + 12$ ✓

$\Rightarrow p(t) = -2t^2 + 12t + D$ ✓

$80 = p(0) = 0 + 0 + D \Rightarrow D = 80$ ✓

✓ $p(t) = -2t^2 + 12t + 80 = -2(t^2 - 6t - 40)$

Want t where $p(t) = 0$, ✓

$0 = -2(t-10)(t+4)$

not in domain

$\Rightarrow t = 10$ or $t = -4$

$\Rightarrow$ $\boxed{10 \text{ seconds}}$ ✓

2. (12 points) Calculate the following limits. If the limit does not exist, explain why.

(a) $\lim_{x\to 0^-} \frac{e^x}{x^3 - 2x}$ $= \frac{e^0}{0-0} = \frac{1}{0} \Rightarrow \infty$ or $-\infty$ (4)

as $x \to 0^-$, get $+$ on top

bottom is $x(x^2-2)$ (2)

$\downarrow \quad \downarrow$

$- \cdot - = +$

6

$\Rightarrow \boxed{+\infty}$

+1 for $-\frac{1}{2}$

(b) $\lim_{x\to 0} \frac{\tan(3\pi x)}{\ln(1+x)}$ $\rightsquigarrow \frac{\tan 0}{\ln(1)} = \frac{0}{0}$ ✓

6

$\overset{H}{=} \lim_{x\to 0} \frac{\sec^2(3\pi x)\cdot 3\pi}{\frac{1}{1+x}}$ ✓ ✓ ✓

$= \lim_{x\to 0} \frac{3\pi}{\cos^2(3\pi x)} \cdot \frac{1+x}{1}$ ✓

$= \frac{3\pi}{(-1)^2} = \boxed{3\pi}$ ✓

$-\frac{1}{2}$ for no limit notation.

3. (8 points) For the following statements, answer True or False. If True, give a brief explanation of why. If False, either explain why or give a counterexample. A picture may help, but needs some words to explain it.

(a) "For a continuous function defined on a closed interval, the absolute maximum and absolute minimum values must occur at either the endpoints of the interval or a critical number."

True. This is method for finding abs. max or min (The extreme value thm).

(b) "Suppose I wanted to use Newton's Method on $f(x) = 2x^3 - 24x + 31$. Since $f(2) = -1$, $x_1 = 2$ would be a good first guess."

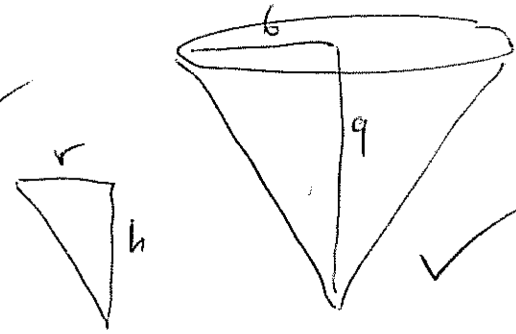
False. Because $f'(x) = 6x^2 - 24$

$\Rightarrow f'(x_1) = f'(2) = 24 - 24 = 0$

So Newton's method would not work.

4. (12 points) Water is pouring into an inverted cone-shaped tank. The cone has a radius of 6 meters and a height of 9 meters. Note that the cone is <u>inverted</u>, so the base of the cone is actually the top of the tank. When the depth of the water is 3 meters, the depth is increasing at a rate of 0.3 meters/second. How fast is the volume of the water changing at that instant? (Note: The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, and you will need to use the fact that the radius equals 2/3 times the height).

$r = \frac{2}{3}h$



Know: $\frac{dh}{dt} = 0.3$ when $h = 3$

Want: $\frac{dV}{dt}$

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h$$

$$\frac{d}{dt}\left(V = \frac{4}{27}\pi h^3\right)$$

$$\frac{dV}{dt} = \frac{4}{\cancel{27}\,9}\pi\cdot\cancel{3}h^2\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{4\pi}{9}\cdot h^2\cdot\frac{dh}{dt}$$

$$= \frac{4}{9}\pi(3)^2\cdot 0.3$$

$$= \boxed{1.2\pi}$$

+10.5 if forgot product rule.

5. (6 points) Find the most general antiderivative of

$$f'(x) = \frac{x^8 - 4x^2 + 1}{x^3}$$

$f'(x) = x^5 - 4x^{-1} + x^{-3}$

$f(x) = \boxed{\frac{1}{6}x^6 - 4\ln|x| - \frac{1}{2}x^{-2} + C}$

6. (12 points) Consider the function $f(x) = \dfrac{x}{3x^2+12}$.

(a) Find the x-coordinate(s) where f has a local maximum or a local minimum. Clearly identify if your answer is a local maximum or a local minimum, and show your work.

$$f'(x) = \frac{3x^2+12 - x(6x)}{(3x^2+12)^2} = \frac{-3x^2+12}{(3x^2+12)^2} = \frac{-3(x^2-4)}{(3x^2+12)^2}$$

$$\Rightarrow 0 = \frac{-3(x+2)(x-2)}{(3x^2+12)^2} \quad \text{at } x = \pm 2$$

$v'(0) = +$

$v'(-4) = -$

$v'(4) = -$

- - - - + + + + - - - - - -

-2 2

$\boxed{x = -2 \text{ is a local min}}$

$\boxed{x = 2 \text{ is a local max}}$

(b) Find the absolute maximum of f on the interval $[0, 4]$.

3

$f(0) = 0$

$f(4) = \frac{4}{60} = \frac{1}{15}$

$f(2) = \frac{2}{24} = \boxed{\frac{1}{12}} \rightarrow$ abs. max

7. (10 points) Given the equation $3x^2 - x = 1$, use Newton's Method to find a solution for x. Use a first guess of $x_1 = 1$ and find the solution accurate to the first two decimal places.

$f(x) = 3x^2 - x - 1$ ✓

$f'(x) = 6x - 1$ ✓

$x_1 = 1$ ✓

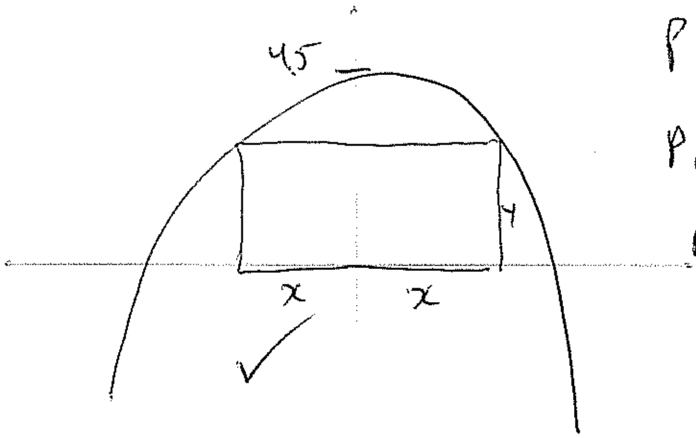
$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{5} = \frac{4}{5}$ ✓✓✓

$x_3 = \frac{4}{5} - \frac{f(0.8)}{f'(0.8)} = 0.7684$ ✓✓

$x_4 = 0.7684 - \frac{f(0.7684)}{f'(0.7684)} = 0.7684$

$\Rightarrow \boxed{0.77}$ ✓

8. (10 points) A rectangle with its bottom on the x-axis has upper corners on the parabola $y = 4.5 - \frac{1}{2}x^2$. Find the maximum perimeter possible for such a rectangle. Be sure to check that your answer is a maximum.



$y = 4.5 - \frac{1}{2}x^2$

$P = 4x + 2y$

$P(x) = 4x + 2(4.5 - \frac{1}{2}x^2)$

$P(x) = 4x + 9 - x^2$

$P'(x) = -2x + 4$

$0 = -2x + 4$

$x = 2$

$\Rightarrow y = 2.5$

$\Rightarrow \boxed{P = 13}$

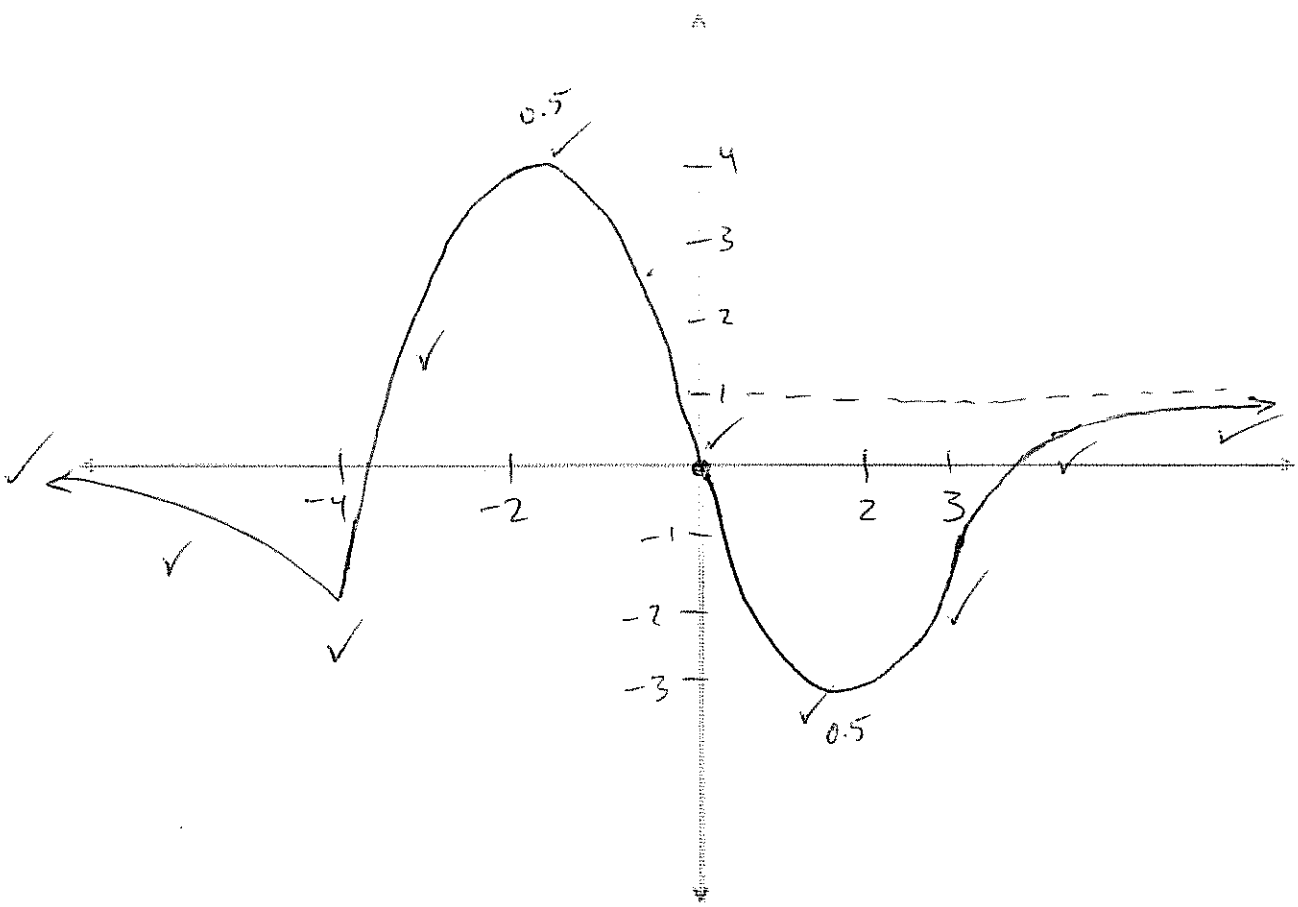
$P''(x) = -2$

$\Rightarrow$ conc. down

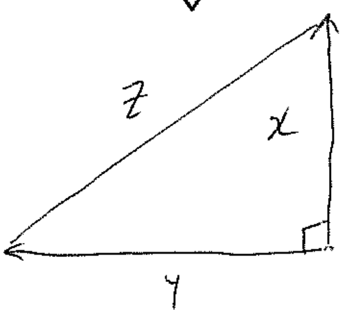
$\Rightarrow$ Max.

9. (12 points) Sketch the graph of a continuous function $f(x)$ given the following information:

- $f(0) = 0$, the absolute minimum value of f is -3, and the absolute maximum value of f is 4.
- $f'(-4)$ and $f'(3)$ do not exist.
- $f'(x) > 0$ on the intervals $(-4,-2), (2,3)$ and $(3,\infty)$.
- $f'(x) < 0$ on the intervals $(-\infty,-4)$ and $(-2,2)$.
- $f''(x) > 0$ on the interval $(0,3)$
- $f''(x) < 0$ on the intervals $(-\infty,-4)$ and $(-4,0)$ and $(3,\infty)$.
- $\lim_{x\to(-\infty)} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$



10. (8 points) Two cars leave from the same point at the same time. One travels north at 36 miles/hour and the other travels west at 27 miles/hour. Use calculus to find the rate at which the two cars are moving away from each other after two hours.



$$\frac{d}{dt}\left(x^2+y^2=z^2\right)$$

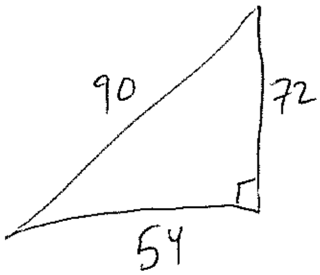
$$\Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$$

$$\frac{dx}{dt}=36$$

$$\frac{dy}{dt}=27$$

Want: $\frac{dz}{dt}$

at two hours:



$$\cancel{2}\cdot 72(36)+\cancel{2}\cdot 54(27)=\cancel{2}\cdot 90\frac{dz}{dt}$$

$$\frac{48\,\cancel{72}\cdot 36+\overset{36}{\cancel{54}}(27)}{5\;\cancel{10}\;\cancel{90}}=\frac{dz}{dt}$$

$$\frac{144+81}{5}=\frac{dz}{dt}$$

$$\boxed{45=\frac{dz}{dt}}$$

Extra Credit(2 points) Find an antiderivative of $f'(x)=e^e$.

$$\boxed{e^e x}$$