2B 2007

PID Number: Key

1. (12 points) A projectile is shot into the air so that its height above the ground after t seconds is given by

$$h(t) = -10t^2 + 220t + 24$$

(a) At what value of t does the projectile reach maximum height?

$h'(t) = -20t + 220$ $\qquad 0 = -20t + 220$

$h''(t) = -20$ $\qquad \frac{-220}{-20} = \frac{-20t}{-20}$ $\qquad 11 = t$

$\Rightarrow$ always concave down

$\boxed{\text{So } h \text{ has a maximum when } t = 11}$

(b) What is the maximum height reached?

$h(11) = -10(11)^2 + 220(11) + 24$

$= -1210 + 2420 + 24$

$= \boxed{1234 \text{ ft}}$

2. (12 points) Given that $A = 1000(1 + \frac{r}{12})^{48}$. This formula for A tells us the dollar value, after 4 years, of an account with an initial deposit of 1000 dollars and interest compounded monthly at the interest rate r. Suppose that the current rate is 5%; that is, $r = 0.05$. Use differentials to approximate the change in value of such an account if r were to increase from its current level to 0.052. Round your final answer to the nearest hundredth.

$A'(r) = 1000 \cdot 48 \left(1 + \frac{r}{12}\right)^{47} \cdot \frac{1}{12} = 4000 \left(1 + \frac{r}{12}\right)^{47}$

$dA = A'(r)\, dr$ $\qquad r = 0.05, \quad dr = 0.052 - 0.05 = 0.002$

$dA = 4000 \left(1 + \frac{0.05}{12}\right)^{47} \cdot 0.002$

$\boxed{dA = 9.73}$

3. (16 points) You are given that $g(x) = \frac{x^2}{3(x^2+27)}$, $g'(x) = \frac{18x}{(x^2+27)^2}$, $g''(x) = \frac{-54(x^2-9)}{(x^2+27)^3}$. Use calculus to answer the following questions:

(a) Does the graph of g have a local maximum or minimum? If so, what are its coordinates?

No discontinuous points

$g'(x) = 0$ when $18x = 0 \Rightarrow x = 0$

at zero, g'' is positive so g has a minimum at $x=0$, the coordinates are $(0,0)$

(b) On what interval(s) is g increasing?

g is increasing when $g'(x)$ is positive. Since the denominator is always positive, g' will be positive when x is positive, on the interval $\boxed{(0, \infty)}$

(c) On what interval(s) (if any) is the graph of g concave downward?

When g'' is negative. $-54(x^2-9) = 0$ when $x = -3$ or 3. Test points $-4, 0, 4$

$g''(-4) =$ negative, $g''(0)$ is positive, $g''(4)$ is negative. Thus g is concave downward on $(-\infty, -3)$ and $(3, \infty)$

(d) On what interval(s) (if any) is the graph of g concave upward?

On the interval $(-3, 3)$ (See above for reasoning)

(e) Find the (x, y)-coordinates of each point of inflection for g. Answers *must* be written as ordered pairs. Inflection points when $g = 3$ and -3

$g(3) = \frac{9}{3 \cdot 36} = \frac{1}{12}$

$(3, 1/12)$

$g(-3) = \frac{(-3)^2}{3((-3)^2+27)} = \frac{9}{3 \cdot 36} = \frac{1}{12}$

$(-3, 1/12)$

4. (12 points) An apple orchard has an average yield of 90 apples per tree if tree density is 20 trees/acre. For each unit increase in tree density (i.e. increase of one tree/acre), the yield decreases by 2 apples. Use calculus to determine how many trees should be planted to maximize the yield. Use the first or second derivative test (whichever you find appropriate) to verify that the apple yield is maximized.

x = increase in tree density above 20

$90-2x$ = yield per tree

Total Yield $= (20+x)(90-2x)$

$= 1800 + 90x - 40x - 2x^2$

$Y(x) = 1800 + 50x - 2x^2$

$Y'(x) = 50 - 4x$

$0 = 50 - 4x \Rightarrow x = \frac{50}{4} = 12.5$

32 trees produces 2112 apples

33 trees produces 2112 apples

12.5 is a critical point

$Y''(x) = -4 \Rightarrow Y$ is always concave down, so $x = 12.5$ is a maximum.

So $20 + 12.5 = 32.5$ trees should be planted (ie., either 32 or 33) ~~$Y(12.5) = (20+12.5)(90-2(12.5))$~~ ~~2112.5~~

5. (12 points) Find all **vertical and horizontal** asymptotes, if any, for the given functions:

(a) $f(x) = \frac{x^3}{x+2}$ $\rightarrow x^3 = 0 \Rightarrow x = 0$, $\rightarrow x + 2 = 0 \Rightarrow x = -2$

So there is a vertical asymptote at $x = -2$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{(x+2)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x^2}{1+\frac{2}{x}} = \lim_{x\to\infty} x^2 = \infty$

$\lim_{x\to-\infty} f(x)$. Same steps as ↗, get ∞.

So no horizontal asymptotes.

(b) $f(x) = \frac{6x^2+30x}{5x^2-125}$ $\rightarrow$ ~~$\frac{x(x+5)}{5(x^2-25)}$~~ $= \frac{6x(x+5)}{5(x^2-25)} = \frac{6x(x+5)}{5(x-5)(x+5)}$

top = 0 when $x = 0$ or $x = -5$

bottom = 0 when $x = 5$ or $x = -5$

So f has a vertical asymptote at $x = 5$ only

$\lim_{x\to\infty} \frac{6x^2+30x}{5x^2-125} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{6+\frac{30}{x}}{5-\frac{125}{x^2}} = \frac{6}{5}$

Same for $-\infty$. Thus $y = 6/5$ is a horizontal asymptote.

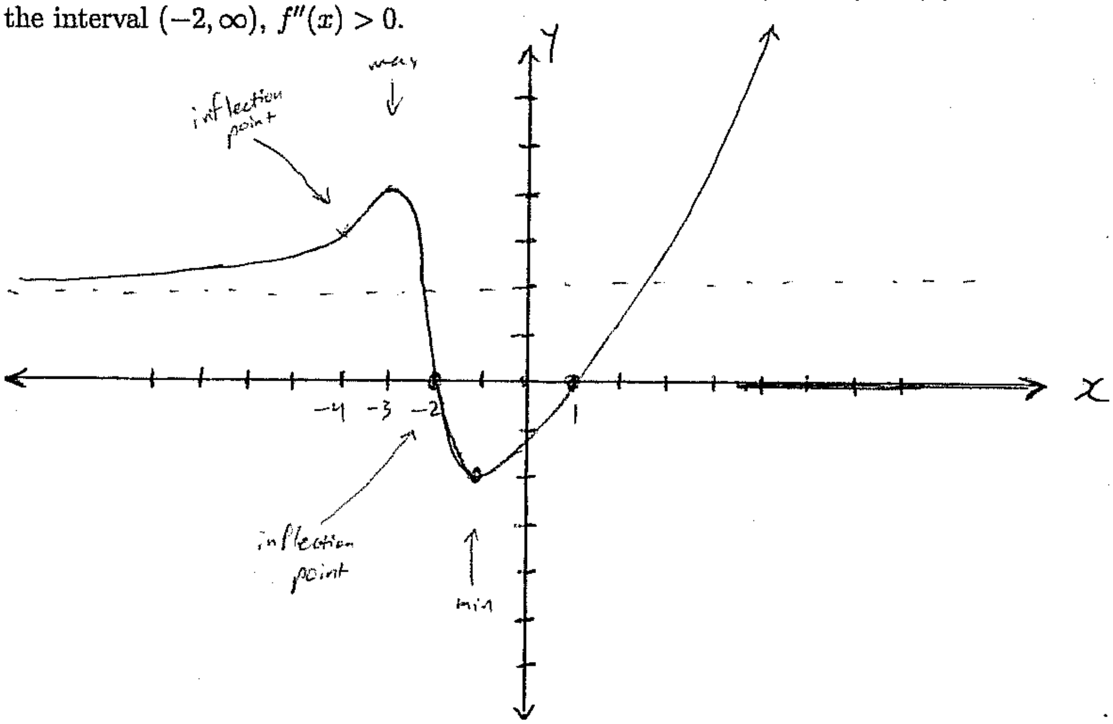
6. (12 points) Sketch the graph of a continuous function f that has all of the following characteristics:

(a) $(-2, 0)$ and $(1, 0)$ are the only x-intercepts

(b) $\lim_{x \to -\infty} f(x) = 2$ and $\lim_{x \to \infty} f(x) = \infty$

(c) On the interval $(-\infty, -3)$, $f'(x) > 0$. On the interval $(-3, -1)$, $f'(x) < 0$. On the interval $(-1, \infty)$, $f'(x) > 0$

(d) On the interval $(-\infty, -4)$, $f''(x) > 0$. On the interval $(-4, -2)$, $f''(x) < 0$. On the interval $(-2, \infty)$, $f''(x) > 0$.



7. (12 points) Use calculus to find the absolute maximum and absolute minimum values of $F(t) = t^3 - 243t$ on the interval $[-5, 15]$.

absolute maximum is 1090 when $t =$ -5

absolute minimum is -1458 when $t =$ 9

$F'(t) = 3t^2 - 243$

$0 = 3(t^2 - 81) = 3(t-9)(t+9)$

critical points at $t = 9, -9$ ← not in our interval

check $F(-5)$ $= -125 + 1215$ $= 1090$ | $F(15)$ $= -270$ | $F(9)$ $= -1458$

8. (12 points) The distance $d(t)$ covered by a car after t seconds is given by

$$d(t) = -2t^3 + 18t^2 - 4t.$$

At what value of t does the car change from accelerating to decelerating?

$d'(t) = -6t^2 + 36t - 4$

$d''(t) = -12t + 36$

$0 = -12t + 36$

$12t = 36$

$\boxed{t = 3}$

On $(0,3)$ $d''(t)$ is positive, so the car is accelerating. On $(3, \infty)$ $d''(t)$ is negative, so the car is decelerating.