Test 3 - Abstract Algebra

Dr. Graham-Squire, Spring 2016

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

1. Don't panic.
2. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
3. Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul.
4. If you are confused about what a particular notation means (e.g. $U(n)$) or whether or not something can be assumed (as opposed to needing to prove it), feel free to ask. Note: S_n is the group of permutations of the numbers 1 through n, D_m is the group of symmetries of a regular m-sided figure, and $U(k)$ is the group of positive integers less than k which are relatively prime to k, under multiplication
5. You must to all of the first four questions, but only two of the last three (if you do all of the last three questions, I will grade them all and give you the two highest scores of the three).
6. Make sure you sign the pledge above.
7. Number of questions = 6. Total Points = 30.

1. (5 points) Let C be the set of all continuous real-valued functions (that is, functions $f: \mathbb{R} \to \mathbb{R}$) whose graphs pass through the point $(1,0)$ (that is, $f(1) = 0$ for all $f \in C$). For the following questions, you must make sure the underlined condition holds for each of the questions.

(a) Prove that C is a ring by listing out the conditions for C to be a ring, and checking that the underlined condition above holds.

(b) Is C commutative? Explain why or why not.

(c) Does C have a unity? If so, what is it?

(a). $f(1)+g(1) = 0+0 = g(1)+f(1) \Rightarrow g+f = f+g$ ✓ Abelian

$\sqrt{}\sqrt{}$ for list

- $(f(1)+g(1))+h(1) = (0+0)+0 = 0 = 0+(0+0) = f(1)+(g(1)+h(1))$ ✓ Assoc.
- Inverse: $\forall f \in C$, $\exists -f$ s.t. $-f(x)+f(x) = 0$ $\forall x$. since $f(1)=0$, have $-f(1) = -0 = 0$ also $\Rightarrow -f \in C$.

$\sqrt{}$ for Answer

- $f(x) = 0$ is identity, and $f(1)=0 \Rightarrow 0 \in C$.
- Assoc. for mult: $(f(1)\cdot g(1))h(1) = 0 = f(1)(g(1)\cdot h(1))$
- $f(g+h) = fg+fh \Leftarrow f(1)(g(1)+h(1)) = 0(0+0) = 0\cdot 0+0\cdot 0$
$= f(1)g(1)+f(1)h(1)$

✓ (b) C is commutative. $fg = gf$, and since $f(1)=g(1)=0$, it doesn't matter what order you do the mult. in.

✓ (c) No, no unity. Should be $f(x) = 1$, but need $f(1)=0$, and $f(x)=1 \Rightarrow f(1) = 1 \neq 0$.

2. (5 points) Let G be an Abelian group with order 400. What are all the possibilities for the group structure of G (up to isomorphism)? That is, what are all of the possible isomorphism classes of G? Explain your reasoning.

$400 = 4 \cdot 100 = 2^2 \cdot 4 \cdot 25 = 2^4 \cdot 5^2$ ✓ 0.5

Know $G \cong \mathbb{Z}_{p_1^{r_1}} \oplus \mathbb{Z}_{p_2^{r_2}} \ldots$ by Fund. Theorem,

Possibilities for $\mathbb{Z}_{2^4} \Rightarrow \mathbb{Z}_{16}$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ ✓

for $\mathbb{Z}_{5^2} \Rightarrow \mathbb{Z}_{25}$ and $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ ✓

So there are $\underline{10}$ possible isom. classes:

$\mathbb{Z}_{16} \oplus \mathbb{Z}_{25}$

$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}$.

$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{25}$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}$

and

$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$

$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$

$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus$

1.5

Missing 1→2 −0.5

3→4 −1

3. (5 points) Let G and H be groups.

(a) Prove that the mapping $\phi : G \oplus H \to G$ given by

$$\phi(g, h) = g$$

is a homomorphism. (Note: ϕ is often referred to as the *projection* of $G \oplus H$ onto G.)

(b) What is the kernel of ϕ?

3 (a) $\phi\big((a,b)(c,d)\big) = \phi\big((ac, bd)\big) = ac = \phi(a,b)\,\phi(c,d)$

(b) $\ker \phi = \{(g,h) \in G \oplus H \mid \phi(g,h) = e\}$ where e is the identity in G.

Since $\phi(g,h) = g$ $\forall g \in G$, the Kernel of ϕ will be all elements in $G \oplus H$ of the form (e, h). So

$$\boxed{\ker \phi = \{(e,h) \mid h \in H\}}$$

4. (5 points) (a) Prove that $\mathbb{Z}$ is normal in $\mathbb{Q}$ (note: both of those are *additive* groups).

(b) Describe an arbitrary element of $\mathbb{Q}/\mathbb{Z}$, and explain why $\mathbb{Q}/\mathbb{Z}$ is an infinite group (that is, why $|\mathbb{Q}/\mathbb{Z}|$ is infinite).

(c) Explain why every element of $\mathbb{Q}/\mathbb{Z}$ has finite order. (note: an example might help your explanation, but one example is not a sufficient general explanation)

(a) Need to show that $q + z + (-q) \in \mathbb{Z}$ for all

√√ $q \in \mathbb{Q}$ and $z \in \mathbb{Z}$. Since $\mathbb{Q}$ is ~~commutative~~ Abelian, have

$$q + z + (-q) = z + q + (-q) = z + 0 = z \in \mathbb{Z}.$$

(b) An arbitrary element of $\mathbb{Q}/\mathbb{Z}$ looks like the coset

$$\frac{p}{q} + \mathbb{Z} = \{\ldots, \frac{p}{q} - 2, \frac{p}{q} - 1, \frac{p}{q}, \frac{p}{q} + 1, \ldots\}$$

1.5 The elements $\frac{1}{2} + \mathbb{Z}, \frac{1}{3} + \mathbb{Z}, \frac{1}{4} + \mathbb{Z}, \frac{1}{5} + \mathbb{Z}, \ldots, \frac{1}{k} + \mathbb{Z}, \ldots$ will all be different cosets b/c they will each have an element $\frac{1}{m}$ in them, and the denominators will all be different $\Rightarrow$ different cosets

(c) Then element $\frac{p}{q} + \mathbb{Z}$ has finite order b/c

$$q\left(\frac{p}{q} + \mathbb{Z}\right) = \frac{p}{q} \cdot q + \mathbb{Z} = p + \mathbb{Z}$$

1.5 Since $p \in \mathbb{Z}$, $p + \mathbb{Z} = \mathbb{Z}$ so

$\frac{p}{q} + \mathbb{Z}$ has order q, which is finite.

at least 0.5 for trying (b), (c)

For the next three problems, you will receive the highest 2 scores out of the three, so you do NOT have to answer all of them.

5. (5 points) Let $\phi : G \to \overline{G}$ be a homomorphism, with G and $\overline{G}$ finite.

(a) Explain why $|\phi(G)|$ divides both $|G|$ and $|\overline{G}|$.

(b) If $|G|$=100 and $|\overline{G}|$=40, what are all of the possibilities for $|\phi(G)|$?

(a) $\phi(G)$ is a subgroup of $\overline{G}$, so by Lagrange's theorem we have $|\phi(G)|$ divides $|\overline{G}|$ 1.5

1.5 ϕ a homomorphism implies it must be an n-to-1 mapping. Thus $\frac{|G|}{|\phi(G)|} = n$, so $|\phi(G)|$ divides $|G|$.

(b) $|\phi(G)|$ must divide both 100 and 400, so $|\phi(G)|$ could be 20, 10, 5, 4, 2 or 1

✓✓ ⇒ allow one mistake with good exp.

6. (5 points) Let G be the group $U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$ (under multiplication mod 32). Let $H = \{1, 31\}$. The group G/H is isomorphic to an Abelian group of order 8. Which one is it? Explain your reasoning.

Cosets: H

$3H = \{3, 29\}$

$5H = \{5, 27\}$

$7H = \{7, 25\}$

$9H = \{9, 23\}$

$11H = \{11, 21\}$

$13H = \{13, 19\}$

$15H = \{15, 17\}$

$\langle 3H \rangle = \{3H, 9H, 5H, 15H, 13H, \ldots$ $\Rightarrow$ has order 8 b/c > 4

0.5

$11 \cdot 31 = 341$

$341 \bmod 32 = 21$

3H is cyclic $\Rightarrow$ $G/H \approx \mathbb{Z}_8$

0.5

3 for cosets and possibilities, but no proof of how to choose.

7. (5 points) Let a belong to a ring R. Let $S = \{x \in R \,|\, ax = 0\}$ (S is sometimes referred to as the "annihilator" of a). Prove that S is a subring of R.

- S is nonempty b/c $0 \in R$ and $a0 = 0$
- ✓ Closure under subtraction:

  Suppose $x, y \in S$. Then $ax = 0$ and $ay = 0$

  1.5 $a(x-y) = ax - ay = 0 - 0 = 0$

  $\Rightarrow x - y \in S$ ✓

- ✓ Closure under mult.

  $a(xy) = (ax)y = 0y = 0$.

  1.5 $\Rightarrow xy \in S$ ✓

8. Extra Credit: (up to 2 points) Write either 0.5 or 2 points. If you put 0.5, you are guaranteed to get an extra 0.5 points on your test. If you put 2, and less than half the students in the class put 2, then you get your 2 points. If half or more of the students put 2 points, though, then everyone who put 2 gets no extra credit points.

0.5	2
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