Test 2 - Abstract Algebra

Dr. Graham-Squire, Spring 2016

9:20
9:39

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

(*signature*)

DIRECTIONS

1. Don't panic.
2. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
3. Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul.
4. If you are confused about what a particular notation means (e.g. $U(n)$) or whether or not something can be assumed (as opposed to needing to prove it), feel free to ask.
5. You must to all of the first four questions, but only two of the last three (if you do all of the last three questions, I will grade them all and give you the two highest scores of the three).
6. Make sure you sign the pledge above.
7. Number of questions = 6. Total Points = 30.

1. (5 points) Some of the following six groups are isomorphic and others are not isomorphic. If a group is not isomorphic to other groups, give a (brief) explanation of why. If two groups *are* isomorphic, give a brief explanation of why (full proof is not necessary).

$$S_4 \qquad D_{12} \qquad \mathbb{Z}_3 \oplus \mathbb{Z}_8 \qquad \mathbb{Z}_{24} \qquad U(5) \oplus \mathbb{Z}_6 \qquad \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$$

- S_4 and D_{12} are both Non-Abelian, and the other four are Abelian, so S_4 and D_{12} are not isomorphic to the others.

- Max order of an element in S_4 is 4 (only four entries in the permutation) while D_{12} has an element of order 12 (R_{30}), so $S_4 \not\approx D_{12}$

- $\mathbb{Z}_3 \oplus \mathbb{Z}_8 \approx \mathbb{Z}_{24}$ b/c both are cyclic of order 24 (Note that $\gcd(3,8)=1$)

- $U(5) \approx \mathbb{Z}_4 \Rightarrow U(5) \oplus \mathbb{Z}_6 \approx \mathbb{Z}_4 \oplus \mathbb{Z}_6 \not\approx \mathbb{Z}_{24}$ b/c $\gcd(4,6)=2$ and not $\underline{1}$, so max order an element of $U(5) \oplus \mathbb{Z}_{12}$ is 12, so it is not cyclic.

- $|\mathbb{Z}_{12} \oplus \mathbb{Z}_{12}| = 144$ ~~the order~~ and all other groups have order 24, so $\mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$ is not isomorphic to anything on the list.

0.5/1 if recognize S_4, D_{12} are special, but no $\approx$

2. (4 points) Give at least *two* reasons why $\mathbb{Z}$ (under addition) is not isomorphic to $\mathbb{R}$ (under multiplication).

√√. $\mathbb{Z}$ is countably infinite and $\mathbb{R}$ is uncountably infinite, so there is not 1-1 (or onto) mapping.

√√. $\mathbb{Z}$ is cyclic (generated by 1) and $\mathbb{R}$ is <u>not</u>.

- $-1 \in \mathbb{R}$ has order 2 (under mult.) but there is no element in $\mathbb{Z}$ with order 2.

3. (5 points) Let G be the following subgroup of S_6:

$$G = \{(1), (12)(34), (1234)(56), (13)(24), (1432)(56), (56)(13), (14)(23), (24)(56)\}.$$

Recall that the *stabilizer* is the set of elements of the permutation group that send a number to itself, and the *orbit* is all of the numbers that a particular number can get sent to.

(a) Find the stabilizer of 1 (in G) and the orbit of 1 (in G).

(b) Find the stabilizer of 5 (in G) and the orbit of 5 (in G).

(c) In what way(s) do your answers above confirm or refute the orbit-stabilizer theorem?

1.5 (a) $\mathrm{Stab}_G(1) = \{(1), (24)(56)\}$ $\quad$ $\mathrm{orb}_1 G = \{1, 2, 3, 4\}$

1.5 (b) $\mathrm{Stab}_G(5) = \{(1), (12)(34), (13)(24), (14)(23)\}$ $\quad$ $\mathrm{orb}_5 G = \{5, 6\}$

✓✓ (c) For both (a) and (b) $|\mathrm{Stab}| \cdot |\mathrm{orb}| = 8 = |G|$,

so they confirm the orbit-stabilizer Thm.

4. (4 points) Let $G = U(15) \oplus \mathbb{Z}_{30} \oplus S_8$. Find the order of the element

$$(2, 7, (123)(154)) \in G.$$

Explain your reasoning.

1.5 $\left\{ \begin{array}{l} |2| \text{ in } U(15) \text{ is } \underline{4} \text{ b/c } 2^4 = 16 \equiv 1 \pmod{15} \\ |7| \text{ in } \mathbb{Z}_{30} = \underline{\underline{30}} \text{ b/c } 7 \text{ is relatively prime to } 30. \end{array} \right.$

$(123)(154) = (15423)$ and

✓ $|(15423)|$ is $\underline{\underline{\underline{5}}}$ b/c it is a 5-cycle.

So $|(2, 7, (123)(154))| = \text{lcm}(4, 30, 5) = \boxed{60}$ ✓

0.5

0.5/2.5 just for knowing need to get orders (more for work)

For the next three problems, you will receive the highest 2 scores out of the three, so you do NOT have to answer all of them.

5. (6 points) Let H be a subgroup of G. Prove that, for $a \in G$,

$$aH = H \text{ if and only if } a \in H.$$

2.5

($\Leftarrow$) if $a \in H$, then ~~since $e \in H$ (b/c H is a subgroup)~~ ~~we have $a e \in$~~ since H is a subgroup, $ah \in H$ for all $h \in H$. Thus $aH \subseteq H$, and since it is a coset, they must have same size $\Rightarrow aH = H$.

($\Rightarrow$) Suppose $aH = H$. Then since $e \in H$ (b/c H is a subgroup), we have $a \cdot e \in H$. But $a \cdot e = a$, so $a \in H$. $\square$

2.5

1/2.5 for an attempt, not totally wrong but with little correct either.

6. (6 points) Prove the following for groups G and H:

If $G \oplus H$ is cyclic, then G and H are both cyclic.

Suppose $G \oplus H$ is cyclic. ~~Suppo~~ Let $G \oplus H = \langle (a,b) \rangle$, where $a \in G$, $b \in H$, and (a,b) generate $G \oplus H$. In particular, this means that $(g, e) = (a,b)^k$ for any $g \in G$. Thus $(g,e) = (a^k, b^k) \Rightarrow \forall g \in G, \exists k$ s.t. $g = a^k$, so a generates G. Similarly, $(e, h) = (a,b)^j$ for any $h \in H$ and some $j \in \mathbb{Z}$ (b/c (a,b) generates $G \oplus H$), so $h = b^j$ and thus b generates H. Since $G = \langle a \rangle$ and $H = \langle b \rangle$, G and H are both cyclic.

7. (6 points) Suppose $\alpha : G \to H$ is an isomorphism from G to H, and $\beta : H \to K$ is an isomorphism from H to K. Prove that G is isomorphic to K. (Note: you may have to use certain conclusions we have proved previously in this course (and Math Thought). If you are unsure whether you can state something or need to prove it, ask Dr. G-S).

✓

Mapping: Let $\phi : G \to K$ be $\phi(g) = \beta(\alpha(g))$ 0.5

Can also just reference composition $\Rightarrow$ 1-1, onto

1-1: ✓ Let $\phi(g_1) = \phi(g_2)$

$\Rightarrow \beta(\alpha(g_1)) = \beta(\alpha(g_2))$ 0.5

$\Rightarrow \alpha(g_1) = \alpha(g_2)$ b/c β is $\cong \Rightarrow \beta$ is 1-1

$\Rightarrow g_1 = g_2$ " α " " $\Rightarrow \alpha$ " ".

Onto: ✓ Let $k \in K$. Since β is onto (b/c $\cong$), we have that $\exists\, h \in H$ s.t. $\beta(h) = k$. Similarly, since α is onto we have $\exists\, g \in G$ s.t. $h = \alpha(g)$. Thus 0.5

$\beta(\alpha(g)) = k \Longrightarrow \phi(g) = k$ so ϕ is onto.

O.P.: ✓ $\phi(g_1 g_2) = \beta(\alpha(g_1 g_2))$

$= \beta(\alpha(g_1)\alpha(g_2))$ b/c α is O.P. since $\alpha \cong$.

$= \beta(\alpha(g_1))\beta(\alpha(g_2))$ b/c β is O.P. since β is $\cong$

$= \phi(g_1)\ \phi(g_2)$ 0.5

$\square$

8. (2 points) Extra Credit: Recall that $Inn(G)$ denotes the group of *inner* automorphisms of G, that is, automorphisms of the form ϕ_g where $g \in G$ and $\phi_g : G \to G$ is defined by $\phi_g(x) = gxg^{-1}$. Prove that $|Inn(G)| = 1$ if and only if G is Abelian.

- G Abelian $\Rightarrow$ $\phi_g(x) = gxg^{-1} = gg^{-1}x = x \quad \forall g \in G$

  $\Rightarrow$ $\phi_g(x)$ is same as identity map for all g

  $\Rightarrow$ only one ϕ_g $\Rightarrow$ $|Inn(G)| = 1$.

- $|Inn(G)| = 1$ $\Rightarrow$ only one inner automorphism of G. We know one such automorphism, it is $\phi_e(x) = exe^{-1} = x$, $\forall x \in G$, which is the identity automorphism. So this means $\phi_g(x)$ must equal x for <u>all</u> $g \in G$

  $\Rightarrow$ $x = \phi_g(x) = gxg^{-1} \quad \forall x \in G, g \in G$

  $\Rightarrow$ $x = gxg^{-1}$

  $\Rightarrow$ $xg = gx \quad \forall g, x \in G$

  $\Rightarrow$ G is Abelian.