Quiz 6, Abstract Algebra

Dr. Graham-Squire, Spring 2016

12:37
12:43
6
$\Rightarrow$ 20-25 min.

Name: Key

1. (4 points) Let G be an Abelian group and $|G| = 100$.

   (a) What are all possible $\mathbb{Z}_n$ groups (or direct products of $\mathbb{Z}_n$ groups) that G could be isomorphic to?

   (b) If you knew G is not cyclic and has exactly 2 elements of order 4, does that tell you which group G is isomorphic to? If not, which groups could you eliminate? Justify your answer.

(a) $\mathbb{Z}_4 \oplus \mathbb{Z}_{25}$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$

allow one overlap, then -0.5 for each two

(b) Not cyclic $\Rightarrow$ Not $\mathbb{Z}_4 \oplus \mathbb{Z}_{25} \approx \mathbb{Z}_{100}$

2 elements of order 4 $\Rightarrow$ $\boxed{\mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5}$ b/c have no elements of order 4 where has $(1,0,0)$ and $(3,0,0)$ and $|(1,0,0)| = 4$, $|(3,0,0)| = 4$, and all other elements have order two or 5.

2. (3 points) The mapping $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_8$ given by $\phi(x) = 2x$ is a homomorphism.

(a) Is ϕ an onto mapping (that is, are all 8 elements of $\mathbb{Z}_8$ be in the image of ϕ)?

(b) Calculate the order of the kernel (that is, find $|\mathrm{Ker}\phi|$).

(c) If all you knew was $\alpha : G \to H$ was a homomorphism, $|G| = 20$, and $|H| = 8$, could you tell whether or not α was onto? Explain why or why not.

✓ (a) $\phi(x) = 2x$ will be <u>even</u> for all elements in the image of $\phi \Rightarrow \phi$ is <u>not</u> onto (doesn't hit any odd #s)

✓ (b) Since ϕ maps to $\{0,2,4,6\} \subseteq \mathbb{Z}_8$ ($\phi(0)$, $\phi(1)$, $\phi(2)$, $\phi(3)$), we have that $|\phi(G)| = 4$, so $\frac{|\mathbb{Z}_{20}|}{|\phi(G)|} = \frac{20}{4} = 5 \Rightarrow$

ϕ is a 5-to-1 mapping $\Rightarrow |\mathrm{Ker}\,\phi| = 5$

✓ (c) Yes, you would know it <u>cannot</u> be onto

b/c $|\phi(G)| \,\big|\, |\mathbb{Z}_{20}|$ and $|\phi(G)| \,\big|\, |\mathbb{Z}_8|$

$\Rightarrow |\phi(G)| \,\big|\, 20$ and $|\phi(G)| \,\big|\, 8$

$\Rightarrow |\phi(G)|$ divides $\gcd(20,8) = 4$

$\Rightarrow |\phi(G)| \neq 8$ so the image of ϕ cannot be all of $\mathbb{Z}_8 \Rightarrow$ <u>not</u> onto.

3. (3 points) Let G be the group of all polynomial functions with real coefficients, with *addition* as the group operation.

(a) Prove that the derivative mapping $d : G \to G$ given by $d(f) = f'$ is a homomorphism.

(b) What is the kernel of d?

(a) $d(f+g) = f' + g' = d(f) + d(g)$ ✓

(b) Ker(d) is all constant functions because

$d(c) = 0$ and 0 is identity.