Quiz 5, Abstract Algebra

Dr. Graham-Squire, Spring 2016

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Name: Key

1. (3 points) Let H be a subgroup of G, and suppose that H is also a subgroup of $Z(G)$ (where $Z(G)$ is the center of G, that is all $g \in G$ that commute with *everything* in G). Prove that H is normal in G.

H subgroup of $Z(G) \Rightarrow hx = xh$ for all $x \in G$ and $h \in H$.

Then the coset $xH = \{xh \mid h \in H\} = \{hx \mid h \in H\} = Hx$

b/c H is subgroup of the center

Thus for all $x \in G$, $xH = Hx$

$\Rightarrow H \lhd G$.

1.5 for def. of Normal (or N.S.T.)

2. (4 points) (a) Describe the elements of the factor group $\mathbb{Z}_4 \oplus \mathbb{Z}_4/\langle(1,2)\rangle$.

(b) $\mathbb{Z}_4 \oplus \mathbb{Z}_4/\langle(1,2)\rangle$ is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Which one is it? Explain your reasoning.

(a) $\langle(1,2)\rangle = \{(1,2), (2,0), (3,2), (0,0)\} = H$ ✓

So elements of $\mathbb{Z}_4 \oplus \mathbb{Z}_4 / \langle(1,2)\rangle$ are

H

$(0,1)+H = \{(1,3),(2,1), (3,3), (0,1)\}$

$(0,2)+H = \{(1,0), (2,2), (3,0), (0,2)\}$ ✓

$(0,3)+H = \{(1,1), (2,3), (3,1), (0,3)\}$

(b) So $\langle(0,1)+H\rangle = \{(0,1)+H, (0,2)+H, (0,3)+H, H\}$ ✓

generates the factor group $\Rightarrow \approx \boxed{\mathbb{Z}_4}$

✓

3. (3 points) Recall that $\mathbb{R}^*$ is the group with elements $\mathbb{R} - \{0\}$ and binary operation multiplication. Also, let $\mathbb{R}^+$ be the group of positive real numbers.

Prove that $\mathbb{R}^* = \mathbb{R}^+ \times \{-1, 1\}$.

(Recall that $G = H \times K \Leftrightarrow$ ~~H, K are normal in G,~~ $H \cap K = \{e\}$ and $G = HK$)

You can assume $\mathbb{R}^+$ and $\{-1,1\}$

- ~~Since $\mathbb{R}^*$ is Abelian, all subgroups are normal $\Rightarrow \mathbb{R}^+$, $\{-1,1\} = \langle -1 \rangle$ are normal.~~

- $\mathbb{R}^* \cap \{-1,1\} = \{1\} = \{e\}$ b/c $-1 \notin \mathbb{R}^+$ and $1 \in \mathbb{R}^+$ b/c it is positive. ✓

- Let $x \in \mathbb{R}^*$. If $x > 0$, $\underbrace{x}_{\in \mathbb{R}^+} = x \cdot \underbrace{1}_{\in \{-1,1\}}$ ✓

  If $x < 0$ $x = \underbrace{|x|}_{\in \mathbb{R}^+} \cdot \underbrace{(-1)}_{\in \{-1,1\}}$ ✓

So $\mathbb{R}^* = (\mathbb{R}^+)(\{-1,1\})$

$\Rightarrow \mathbb{R}^* = \mathbb{R}^+ \times \{-1,1\}$ ∎