# Quiz 3, Abstract Algebra

Dr. Graham-Squire, Spring 2016

2:12

Name: Key

1. (5 points) Prove that $\mathbb{Z}_n$ is isomorphic to *any* cyclic group of order $n$. Hint: use the fact that the cyclic group can be expressed as $\langle a \rangle$ for some $a$ in the group, an arbitrary element in the group looks like $a^j$.

Let ~~$\phi$~~ $G = \langle a \rangle$ be a cyclic group generated by $a \in G$.

Then $\phi : \mathbb{Z}_n \to G$ defined by $\phi(x) = a^x$ is a mapping from $\mathbb{Z}_n$ to $G$, which we claim is an isomorphism.

1-1: Suppose $\phi(x) = \phi(y)$. Then $a^x = a^y$

$\Rightarrow a^{-x} a^x = a^{-x} a^y$

$\Rightarrow e = a^{y-x}$

Since $0 \le x, y < n$, $-n < y - x < n \Rightarrow y - x = 0 \Rightarrow y = x$ ✓

Onto: Let $b \in G$. Then $b = a^y$ for some $0 \le y < n$ (Since $|a| = n$). Then $y \in \mathbb{Z}_n$ and $\phi(y) = a^y$ ☒

O.P.: Let ~~[illegible]~~ $x, y \in \mathbb{Z}_n$.

Then $\phi(xy) =$ ~~[illegible]~~ $= a^{xy} = a^x a^y = \phi(x)\phi(y)$ ☒

3 for mapping, 1-1, onto, O.P.

0.5 each for right.

Give at least one reason why

2. (2 points) ~~Prove that~~ $\mathbb{Z}$ (the integers) under addition is NOT isomorphic to $\mathbb{Q}$ (the rationals) under addition. Don't need to prove it, just give a reason.

$\mathbb{Z} = \langle 1 \rangle$ is cyclic, but $\mathbb{Q}$ is not cyclic,

because if $\langle \frac{p}{q} \rangle = \mathbb{Q}$, then

+1 for a reasonable reason

+0.5 if semi-coherent, not totally wrong.

3. (3 points) Are the following two groups isomorphic or not? You do not have to give a full proof, but if you think they are isomorphic you should give a good explanation of why you think so. If you think they are NOT isomorphic, you should explain that also. Your explanation can either come from your understanding of the groups themselves, or from the Cayley tables below (preferably both), but it should be justified by something.

The groups are:

(i) $\{(1), (123456)$, ~~$(135)(246), (14)(25)(36), (153)(264), (165432)\}$ (a subgroup of $S_6$) and~~

(ii) $\{R_0, R_{120}, R_{240}, F_{30}, H, F_{150}\}$ (a subgroup of $D_6$. $H$ is the horizontal flip, and $F_{30}$ and $F_{150}$ are the two flips that are 30 degrees off of vertical).

The two groups have the following Cayley tables, with elements in the permutation group listed as $e, a, b, \ldots, f$ (respectively, with $e$ as the identity) and elements in the subgroup of $D_6$ listed as 1, 2, 3, ..., 6 (respectively). Note: There are many ways to make a mapping between the two sets–you should NOT assume that, for example, element $b$ must map to 3 just because they are both the third things on each list.

| | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ |
| $a$ | $a$ | $b$ | $c$ | $d$ | $f$ | $e$ |
| $b$ | $b$ | $c$ | $d$ | $f$ | $e$ | $a$ |
| $c$ | $c$ | $d$ | $f$ | $e$ | $a$ | $b$ |
| $d$ | $d$ | $f$ | $e$ | $a$ | $b$ | $c$ |
| $f$ | $f$ | $e$ | $a$ | $b$ | $c$ | $d$ |

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 3 | 1 | 5 | 6 | 4 |
| 3 | 3 | 1 | 2 | 6 | 4 | 5 |
| 4 | 4 | 6 | 5 | 1 | 3 | 2 |
| 5 | 5 | 4 | 6 | 2 | 1 | 3 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

1.5

• Not isomorphic! By inspection we can see that this group is Abelian (it has symmetry across the diagonal) but this group is not Abelian (e.g $3(4) = 6$ not $4(3) = 5$ and $5 \neq 6$.

• Or, the group of permutations is cyclic, generated by (123456). The subgroup of $D_6$ is not cyclic! The flips ($F_{30}$, $H$, $F_{150}$) have order 2, and the rotations ($R_{120}$, $R_{240}$) have order 3.

+1.5 for good reason