Quiz 2, Abstract Algebra

Dr. Graham-Squire, Spring 2016

1:48
1:56
8.

Name: Key

6 points.

1. The Fundamental Theorem of Cyclic Groups states the following: "Every subgroup of a cyclic group is cyclic. Moreover, if $|\langle a\rangle| = n$, then the order of any subgroup of $\langle a\rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a\rangle$ has exactly one subgroup of order k – namely, $\langle a^{n/k}\rangle$."

   Use an example to verify the Fundamental Theorem of Cyclic Groups–that is, find an example group that illustrates the properties that the Fundamental Theorem is talking about. Your example group should be large enough to be a good illustration, but small enough that it is manageable.

Consider $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ which is cyclic

Subgroups are
$\{0\} = \langle 0\rangle$
$\{0, 6\} = \langle 6\rangle$
$\{0, 4, 8\} = \langle 4\rangle$
$\{0, 3, 6, 9\} = \langle 3\rangle$
$\{0, 2, 4, \ldots, 10\} = \langle 2\rangle$
$\{0, 1, \ldots, 11\} = \langle 1\rangle$

every subgroup is cyclic

Here $|\langle 4\rangle| = 3$ and 3 is a divisor of 12, same for the others. Here $\langle 1\rangle$ has exactly one subgroup of order 6, It is the subgroup $\langle 2\rangle = \langle 1\cdot\frac{12}{6}\rangle = \langle a^{n/k}\rangle$

in the theorem.

2. You have proved previously (in a homework assignment) that the intersection of two subgroups is a subgroup, but what about the ~~intersection~~ union of two subgroups? Prove or disprove the following: "If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G".

2 points

Disprove by counterexample. Consider

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5, 6\}$. The $\underset{\langle 3 \rangle}{\{0, 3\}}$ and $\underset{\langle 2 \rangle}{\{0, 2, 4\}}$ and

subgroups of $\mathbb{Z}_6$, but $\langle 3 \rangle \cup \langle 2 \rangle = \{0, 2, 3, 4\}$ is <u>not</u>

a subgroup because it is not closed $(2+3=5 \notin \langle 3 \rangle \cup \langle 2 \rangle)$

3. For any element p in a group G, prove that $\langle p \rangle$ is a subgroup of the centralizer of p (recall that the centralizer of p, $C(p)$, is the set of all elements in G that commute with p).

2 points.

Two step: $p \in \langle p \rangle$, so $\langle p \rangle$ is nonempty.

Note that $p^n(p) = p(p^n) = p^{n+1}$ for all n, so $\langle p \rangle \subset C(p)$ for all p.

Let $a, b \in \langle p \rangle$, then $a = p^n$, $b = p^m$ for some $m, n \in \mathbb{Z}$.

Then $ab = p^{n+m} \in \langle p \rangle$ so $\langle p \rangle$ is closed.

Let $a \in \langle p \rangle$ then $a = p^n$ for some $n \in \mathbb{Z}$. Let $b = p^{-n} \in \langle p \rangle$.

The $ab = p^n p^{-n} = p^0 = e \Rightarrow b = a^{-1}$, so $a^{-1} \in \langle p \rangle$. ☒

(could also just point out that $\langle p \rangle \subset C(p)$, and $\langle p \rangle$ is a subgroup by previous theorem proved in class.)