Project 2 - Hyperbolic Functions

MTH 1410 - Dr. Graham-Squire

For this project, you can work by yourself or with a group of up to 3 people, although question 6 must be answered individually even if you work with a group. The project is modified from the discovery project on page 227 in the textbook. You will have about 3 weeks to complete the assignment, though certain parts must be turned in at one week intervals. Specifically, the following due dates will hold:

- Wednesday, Feb 27: Your group must turn in solutions to two of the assigned problems. I will correct them and return them to the group. Your score for this part will be based on both completion and accuracy, in the following manner- One question will be graded on both accuracy and completion, so you should choose at least one question that you understand well to turn in. The other question will be graded just on completion, not accuracy, so you should choose a question that you are not too sure of to turn in for that question. Thus you will be able to get feedback, but not lose points if you did it wrong. Make sure you specify which question you want corrected for accuracy, or I will simply choose the first one to grade for accuracy.
- Friday, Mar. 15: Complete project, both group and individual, due at the beginning of class.

Each individual in the group needs to discover and prove an identity of their own for question number 6 of the project. It is okay to talk with other people in the group to come up with an identity, but the work of proving it must be done by the individual. When the final project is turned in, each member of the group should write up a roughly equal number of questions (i.e. if you are a group of three and there are nine questions, each group member should write up three of the questions. The group should work together to come up with the answer, but the actual write-ups cannot all be done by the same person).

Score Breakdown:

- Part One, due Wednesday, Feb. 27: 20%, grade is based on completion and accuracy as stated above.
- Corrections to part 1: 10%, to be turned in with completed project.
- Part Two, due Friday, Mar. 15: 70%, Group portion (that is, all of the assigned problems along with corrections for the two previously graded)- 60%; Individual portion- 10%. (Graded for both completion and accuracy)

Certain combinations of e^x and e^{-x} are so common in mathematics that we have given them special names, the *hyperbolic functions*. Although they may look very different, the hyperbolic functions have many properties, identities and derivatives that mimic those of the trigonometric functions, and that is what we will explore in this project. In particular, we will look at the following functions, known as the hyperbolic sine, hyperbolic cosine, hyperbolic tangent, and hyperbolic secant functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \text{sech } x = \frac{1}{\cosh x}$$

The reason for the name "hyperbolic" functions is that they are related to the hyperbola in a similar way that the trigonometric functions are related to the circle. One application of these functions is that $\cosh x$ can be used to model a *catenary*, which is the shape that a heavy flexible cable makes when it is stretched across a long distance (think of telephone and electrical wires). You will look at such a graph in question 2.

Assigned Problems:

Make sure to show your work in order to receive full points.

(1) (a) On the set of axes, sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$, for x-values between -4 and 4. On that same set of axes, use graphical addition to sketch a graph of the function $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$. "Graphical addition" just means that you add the y-values together for your two graphs to get a y-value for a third graph.

(b) Check the accuracy of your graph by graphing $y = \cosh x$ on a graphing calculator (you do not need to show any work for this). What are the domain and range of $y = \cosh x$?

(2) Graph four members of the family of curves $y = a\cosh(x/a)$ (Note- this graph is called a *catenary*, and should roughly look like a heavy wire stretched between two poles). That is, choose four different numbers for a (don't choose zero or 1, and at least one of your choices should be negative)and graph the resulting functions. How does the graph change as a varies? Explain why this happens.

(3) An even function has reflectional symmetry across the y-axis, and an odd function will have rotational symmetry about the origin. This means that an *even* function will look like a mirror image if you flipped it over the y-axis and an *odd* function would look the same if you spun it 180 degrees around the point (0,0).

(a) Graph sinh and tanh. From looking at the graphs, guess whether cosh, sinh and tanh are even, odd or neither. That is, first tell me if cosh is even, odd, or neither, then the same for sinh and tanh.

(b) Prove your assertions in part (a) by using the following definitions: A function f is even if $f(x) = f(-x)$ for all values of x. A function is odd if $f(-x) = -f(x)$ for all values of x. For example, the function $f(x) = x^3$ is an odd function because

$$f(-x) = (-x)^3 = (-x)(-x)(-x) = -x^3 = -f(x).$$

You need to give a similar justification for cosh, sinh and tanh.

(4) Prove the identity $\cosh^2 x - \sinh^2 x = 1$.

(5) Prove the identity $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.

(6) **Individual Question**: Questions 4 and 5 are similar to well-known trigonometric identities. Try to discover another identity for the hyperbolic functions (similar to sum, difference, double or half-angle formulas for trigonometry. If you don't know what these are, try googling "trigonometric identities". Note that your hyperbolic identity may have a negative where a trig identity has a positive, or vice versa), and then you **must also prove that it is true**. Each member of the group must do a different one.

(7) The differentiation formulas for the hyperbolic functions are similar to those of the trigonometric functions, though the signs are sometimes different.

(a) Prove that $\frac{d}{dx}(\sinh x) = \cosh x$.

(b) Find the derivatives of $\cosh x$ and $\tanh x$, and prove that they are correct.

(8) (a) We say that a function is *one-to-one* if it is always increasing or always decreasing. Use calculus to explain why sinh is a one-to-one function (just saying "look at the graph" is not enough! You need to talk about slopes, derivatives and their signs).

(b) Find a formula for the derivative of the inverse hyperbolic sine function $y = \sinh^{-1} x$. You should follow the same method using implicit differentiation that we used in the Section 3.6 notes to find the derivative of $y = \cos^{-1} x$. When we found the derivative of arccos, though, we used a triangle to help us and we cannot do that here. You will need to use the fact (derived from question 4) that $\cosh y = \sqrt{1 + \sinh^2 y}$ in order to write the final answer in terms of x.

(c) Use the fact that $\sinh x$ and $\sinh^{-1} x$ are inverse functions to verify that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

To do this, you need to show that $\sinh(\sinh^{-1} x) = x$. There may be other ways to verify the equality, but I expect you to prove that $\sinh(\sinh^{-1} x) = x$.

(d) Take the derivative of $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, and show your work. Compare your answer to what you got in part (b).

(9) At what point on the curve $y = \cosh x$ does the tangent have a slope of 1? Show your work.

Additional Comments:

- You can come to my office hours if you need help or have questions. If you are working with a group, though, you should all come together as a group (and if that means you need to schedule a time to meet with me that is not at my normal office hours time, that is fine).

- If you are working in a group, I expect each member of the group to write up approximately the same amount of the solutions that are turned in at the end. The individual portions, obviously, should be in your own handwriting.
- Late projects will lose 10 percent each day they are late. Projects turned in more than four days late will not be accepted.
- If you are working in a group, it is up to you to decide how you will get the work done. If you divide up the problems, it is important that you check each others work so that you make sure everything turned in is correct.